> restart;

> with(Riemann):with(Canon):

```
> with(TensorPack) : CDF(0) : CDS(index) :
```

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Tensor analysis using indices - Senovilla et al. - Shearfree for dust
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if $\sigma_{a\,b}$=0 =>ωΘ=0
Author: Peter Huf
file 6 - eqs49 & 50

```
> read "EFE" : read "SFE" :read "fids" :read "eqs2" :read "Seneqs5" :
```

****

Equations 49 and 50

****

>

case 2:
if $\omega^{b}\theta_{;b}=0$ then eq44 leads to (with identities (14))

```
> temp := eq[44] : T(%);
```

$$\left(-\frac{1}{4}\theta-\theta k\right)\omega_{a}{}^{d}\omega_{d}{}^{c}\omega_{c}{}^{b}\theta_{;b}+\left(\frac{1}{3}\theta^{2}-\frac{1}{4}\mu+\omega^{2}\right)\omega_{a}{}^{c}\omega_{c}{}^{b}\theta_{;b}+\left(\frac{1}{2}\theta\mu-\frac{8}{3}\theta\omega^{2}\right)\omega_{a}{}^{b}\theta_{;b}+\left(\frac{32}{3}\omega^{4}+\frac{1}{2}\mu^{2}+\frac{32}{9}\theta^{2}\omega^{2}-\frac{14}{3}\omega^{2}\mu\right)\theta_{;b}P_{a}{}^{b}=0 \quad (1.1)$$

```
> temp1 := eq[14 a] : T(%);
```

$$\omega_{a}{}^{c}\omega_{c}{}^{b}=-\omega^{2}P_{a}{}^{b}+\omega_{a}\omega^{b} \quad (1.2)$$

```
> temp2 := eq[14 b] : T(%);
```

$$\omega_{a}{}^{c}\omega_{c}{}^{d}\omega_{d}{}^{b}=-\omega^{2}\omega_{a}{}^{b} \quad (1.3)$$

```
> temp3 := subs(d = e, c = d, e = c, temp2) : T(%);
```

$$\omega_{a}{}^{d}\omega_{d}{}^{c}\omega_{c}{}^{b}=-\omega^{2}\omega_{a}{}^{b} \quad (1.4)$$

```
> temp4 := collect(TEDS(temp3, temp), [P[-a, b], omega[-a, b], theta[-B]], `distributed`) :
    T(%);
```

(1.5)

$$\left(\frac{32}{3}\omega^4+\frac{1}{2}\mu^2+\frac{32}{9}\theta^2\omega^2-\frac{14}{3}\omega^2\mu\right)\theta_{;b}P_a{}^b-\frac{1}{12}\left(-12\omega^2-4\theta^2+3\mu\right)\omega_a{}^c\theta_{;b}\omega_c{}^b+\left(-\frac{29}{12}\theta\omega^2+\omega^2\theta k+\frac{1}{2}\theta\mu\right)\omega_a{}^b\theta_{;b}=0 \quad \textbf{(1.5)}$$

```
> temp5 := collect(TEDS(temp1, temp4), [P[-a, b], omega[-a, b], theta[-B], omega[-a], omega[b]], `distributed`) : T(%);
```

$$\left(\frac{1}{3}\theta^2-\frac{1}{4}\mu+\omega^2\right)\theta_{;b}\omega_a\omega^b+\left(\frac{29}{3}\omega^4+\frac{1}{2}\mu^2+\frac{29}{9}\theta^2\omega^2-\frac{53}{12}\omega^2\mu\right)\theta_{;b}P_a{}^b+\left(-\frac{29}{12}\theta\omega^2+\omega^2\theta k+\frac{1}{2}\theta\mu\right)\omega_a{}^b\theta_{;b}=0 \quad \textbf{(1.6)}$$

```
> temp6 := collect(TEDS(omega[b]·theta[-B] = 0, temp5), [P[-a, b], omega[-a, b], theta[-B]], `distributed`) : T(%);
```

$$\left(\frac{29}{3}\omega^4+\frac{1}{2}\mu^2+\frac{29}{9}\theta^2\omega^2-\frac{53}{12}\omega^2\mu\right)\theta_{;b}P_a{}^b+\left(-\frac{29}{12}\theta\omega^2+\omega^2\theta k+\frac{1}{2}\theta\mu\right)\omega_a{}^b\theta_{;b}=0 \quad \textbf{(1.7)}$$

or, re-arranged:

```
> temp7 := -1·(op(1, op(1, temp6)) =-op(2, op(1, temp6))) : T(%)
```

$$-\left(\frac{29}{3}\omega^4+\frac{1}{2}\mu^2+\frac{29}{9}\theta^2\omega^2-\frac{53}{12}\omega^2\mu\right)\theta_{;b}P_a{}^b=\left(-\frac{29}{12}\theta\omega^2+\omega^2\theta k+\frac{1}{2}\theta\mu\right)\omega_a{}^b\theta_{;b} \quad \textbf{(1.8)}$$

```
> eq[49] := temp7 : T(%);
```

$$-\left(\frac{29}{3}\omega^4+\frac{1}{2}\mu^2+\frac{29}{9}\theta^2\omega^2-\frac{53}{12}\omega^2\mu\right)\theta_{;b}P_a{}^b=\left(-\frac{29}{12}\theta\omega^2+\omega^2\theta k+\frac{1}{2}\theta\mu\right)\omega_a{}^b\theta_{;b} \quad \textbf{(1.9)}$$

Now eq50:

```
> eq[50] := omega[-a, b]·theta[-B] = omega[-a, c]·P[c, b]·theta[-B] : T(%);
```

$$\omega_a{}^b\theta_{;b}=\omega_a{}^cP^{cb}\theta_{;b} \quad \textbf{(1.10)}$$

is easily shown, and from the definition of omega[a,b]

```
> eq[50] := omega[-a, b]·theta[-B] = eta[-a,-c,-d,-e]·omega[d]·u[e]·P[c, b]·theta[-B] : T(%);
```

$$\omega_a{}^b\theta_{;b}=\eta_{acde}\omega^du^eP^{cb}\theta_{;b} \quad \textbf{(1.11)}$$

the vectors $\omega_a{}^b\theta_{;b}$ **and** $P^{cb}\theta_{;b}$ *are orthogonal.*

```
>
```

****

Conclusion:

the two vectors in eq49 are orthogonal due to eq 50 (above), and hence the terms in eq49 must

vanish. Looking at the rhs of eq49 (temp7) the 3 possibilities are:
1. $\theta = 0$ (*completing the proof*)
2. the bracket vanishes and the lemma 3 the proof is finished
3.
$\omega_a{}^b \theta_{;b} = 0$ *means either* **by** *eq50 that either* $P^{cb} \theta_{;b} = 0$ **or** $\omega^d = 0$. *If the latter*, **then** $\omega = 0$ **or** **by** *lemma* 1, $\theta\omega = 0$

>

theorem is proven

>

Hence by argument this leads to $\omega\theta = 0$

> eq[51] := omega·theta = 0 : T(%);

$$\theta\,\omega = 0 \qquad \textbf{(1.12)}$$

> **save** eq, "Seneqs6" :

> PrintSubArray(eq, 1, 51, y)

$$1,\ T_{ab} = \rho\, u_a u_b$$

$$2,\ P_{ab} = u\, u_{ab} + g_{ab}$$

$$3,\ P^a{}_b u^b = 0$$

$$4,\ dX^a = u^b X^a{}_{;b}$$

$$5,\ du^a = u^b u^a{}_{;b}$$

$$6,\ u_{a;b} = \frac{1}{3}\theta P_{ab} + \sigma_{ab} + \omega_{ab} - du_a u_b$$

$$7,\ \theta = u^a{}_{;a}$$

$$8,\ \sigma_{ab} = \frac{1}{2} P_a{}^c P_b{}^d u_{c;d} + \frac{1}{2} P_b{}^c P_a{}^d u_{c;d} - \frac{1}{3}\theta P_{ab}$$

$$9,\ \omega_{ab} = \frac{1}{2} P_a{}^c P_b{}^d u_{c;d} - \frac{1}{2} P_b{}^c P_a{}^d u_{c;d}$$

$$10,\ \omega^a = \frac{1}{2}\eta^{abcd} u_b \omega_{cd}$$

$$11,\ \omega_{ab} = \eta_{abef}\,\omega^e u^f$$

$$12,\ \omega^2 = \frac{1}{2}\omega^{ab}\omega_{ab}$$

13, "iff(iff(omega[-a,-b] = 0,omega[-a]),omega = 0)"

$$14,\ \omega_a{}^c \omega_c{}^b = -\omega^2 P_a{}^b + \omega_a \omega^b$$

$$15,\ \frac{1}{2} u_{b;a} - \frac{1}{2} u_{a;b} = \frac{1}{2} du_a u_b - \frac{1}{2} du_b u_a + \omega^{ab}$$

$$16, \frac{1}{6} u_c u_{b;a} - \frac{1}{6} u_c u_{a;b} + \frac{1}{6} u_b u_{a;c} - \frac{1}{6} u_b u_{c;a} + \frac{1}{6} u_a u_{c;b} - \frac{1}{6} u_a u_{b;c} = 0$$

$$17, \sigma_{ab} = 0$$

$$18, u_{a;b} = \frac{1}{3} \theta P_{ab} + \omega_{ab}$$

$$19, u^{a}{}_{;c;d} - u^{a}{}_{;d;c} = R^{a}{}_{bcd} u^{b}$$

$$20, dottheta + \frac{1}{3} \theta^2 - 2 \omega^2 + \frac{1}{2} \mu = 0$$

$$21, P_a{}^c P_b{}^d \omega_{cd;f} u^f + \frac{2}{3} \theta \omega_{ab} = 0$$

$$22, \omega_a \omega_b - \frac{1}{3} P_{ab} \omega^2 + E_{ab} = 0$$

$$23, E_{ab} = C_{abcd} u^c u^d$$

$$24, H_{ab} = \frac{1}{2} \eta_{ae}{}^{cd} C_{cdbf} u^e u^f$$

$$25, P^{a}{}_{b} \omega^{b}{}_{;f} u^f + \frac{2}{3} \theta \omega^a = 0$$

$$26, 2 P^{ab} \theta_{;b} + 3 P^{a}{}_{b} \omega^{bd}{}_{;d} = 0$$

$$27, \omega^{a}{}_{;a} = 0$$

$$28, H_{ab} = \frac{1}{2} P_a{}^c P_b{}^d \omega^{d;c} + \frac{1}{2} P_b{}^c P_a{}^d \omega^{d;c}$$

$$29, \omega_{ab} \omega^{bc}{}_{;c} = P_a{}^b \omega^c \omega_{b;c} - P_a{}^b \omega^c \omega_{c;b}$$

$$30, \mu \theta + dotmu = 0$$

$$31, (\mu + p)\, du^a + P^{ab} p_{;b}$$

$$32, du^a = 0$$

$$33, u_a = -\frac{f_{;a}}{fdot}$$

$$34, \mu = (c1 - 1)\, p + c2\, \omega^2$$

$$35, dotomega_{ab} = -\frac{2}{3} \theta \omega_{ab}$$

$$36, dotomega = -\frac{2}{3} \theta \omega$$

$$37, \theta \left(c1p - \frac{1}{3} c2\, \omega^2 \right) = 0$$

$$38,\ \frac{\partial}{\partial t}\left(P^{a\ b} f_{;b}\right) = P^{a\ b} f{`}{'}{`}_{;b} + \omega^{a\ b} f_{;b} - \frac{1}{3}\,\theta\, P^{a\ b} f_{;b}$$

$$39,\ -3\, P_a^{\ \ b}\,\omega^{\ c}\,\omega_{b\,;c} - 13\, P_a^{\ \ b}\,\omega^{\ c}\,\omega_{c\,;b} + 2\, P_a^{\ \ b}\,\mu_{;b} = 0$$

$$40,\ -8\,\omega\, P_a^{\ \ b}\,\omega_{;b} + P_a^{\ \ b}\,\mu_{;b} + \omega_a^{\ \ b}\,\theta_{;b} = 0$$

$$41,\ -8\,\omega_a^{\ \ b}\,\omega\,\omega_{;b} + \omega_a^{\ \ c}\,\omega_c^{\ \ b}\,\theta_{;b} + \omega_a^{\ \ b}\,\mu_{;b} = 0$$

$$42,\ P_a^{\ \ b}\,\theta_{;b}\,\mu - \frac{16}{3}\, P_a^{\ \ b}\,\theta_{;b}\,\omega^2 = \frac{1}{2}\,\omega_a^{\ \ c}\,\omega_c^{\ \ b}\,\theta_{;b} + \frac{1}{3}\,\theta\, P_a^{\ \ b}\,\mu_{;b}$$

$$43,\ \frac{1}{2}\left(\mu + p - \frac{16}{3}\,\omega^2\right)\omega_a^{\ \ b}\,\theta_{;b} + \theta\left(\frac{112}{9}\,\omega^2 - \frac{5}{3}\,\mu - \frac{5}{3}\,p\right) P_a^{\ \ b}\,\theta_{;b} + \left(\frac{5}{9}\,\theta^2 - \frac{2}{3}\,\omega^2 + \frac{1}{6}\,\mu + \frac{1}{2}\,p\right) P_a^{\ \ b}\,\mu_{;b} + \frac{7}{6}\,\theta\,\omega_a^{\ \ c}\,\omega_c^{\ \ b}\,\theta_{;b} - \left(\frac{1}{4} + k\right)\omega_a^{\ \ d}\,\omega_d^{\ \ c}\,\omega_c^{\ \ b}\,\theta_{;b} = 0$$

$$44,\ \left(-\frac{1}{4}\,\theta - \theta\, k\right)\omega_a^{\ \ d}\,\omega_d^{\ \ c}\,\omega_c^{\ \ b}\,\theta_{;b} + \left(\frac{1}{3}\,\theta^2 - \frac{1}{4}\,\mu + \omega^2\right)\omega_a^{\ \ c}\,\omega_c^{\ \ b}\,\theta_{;b} + \left(\frac{1}{2}\,\theta\,\mu - \frac{8}{3}\,\theta\,\omega^2\right)\omega_a^{\ \ b}\,\theta_{;b} + \left(\frac{32}{3}\,\omega^4 + \frac{1}{2}\,\mu^2 + \frac{32}{9}\,\theta^2\,\omega^2 - \frac{14}{3}\,\omega^2\,\mu\right)\theta_{;b}\, P_a^{\ \ b} = 0$$

$$45,\ \left(\frac{32}{3}\,\omega^4 + \frac{1}{2}\,\mu^2 + \frac{32}{9}\,\theta^2\,\omega^2 - \frac{14}{3}\,\omega^2\,\mu\right)\omega^{\ b}\,\theta_{;b} = 0$$

$$46,\ \frac{32}{3}\,\omega^4 + \frac{1}{2}\,\mu^2 + \frac{32}{9}\,\theta^2\,\omega^2 - \frac{14}{3}\,\omega^2\,\mu = 0$$

$$47,\ \frac{64}{3}\,\omega^4 = -\mu^2 - \frac{64}{9}\,\theta^2\,\omega^2 + \frac{28}{3}\,\omega^2\,\mu$$

$$48,\ \left(\mu + \frac{7}{3}\,p - \frac{40}{9}\,\omega^2\right)\omega^2 = 0$$

$$49,\ -\left(\frac{29}{3}\,\omega^4 + \frac{1}{2}\,\mu^2 + \frac{29}{9}\,\theta^2\,\omega^2 - \frac{53}{12}\,\omega^2\,\mu\right)\theta_{;b}\, P_a^{\ \ b} = \left(-\frac{29}{12}\,\theta\,\omega^2 + \omega^2\,\theta\, k + \frac{1}{2}\,\theta\,\mu\right)\omega_a^{\ \ b}\,\theta_{;b}$$

$$50,\ \omega_a^{\ \ b}\,\theta_{;b} = \eta_{a\,c\,d\,e}\,\omega^{\ d}\, u^{\ e}\, P^{c\ b}\,\theta_{;b}$$

$$51,\ \theta\,\omega = 0 \tag{1.13}$$

>