```
> restart;
> with(Riemann):with(Canon):
> with(TensorPack) : CDF(0) : CDS(index) :
```

Chapter XX
Tensor analysis using indices - Senovilla et al. - Shearfree for dust
page 3

if $\sigma_{a\,b}$=0 =>$\omega\Theta$=0
Author: Peter Huf
file 4-eq43
-using Sopuerta equations - correct version

In this file we continue to follow the equations outlined by Senovilla et al. (2007) with the assumptions for dust
i.e

```
> read "EFE" : read "SFE" :read "fids" :read "eqs2" :read "Seneqs3d" :
```

****

Equation 43

****

The original eq43 is

```
> eq[42];
```

$$\omega_{-a,\,b}\,dottheta_{-B} + 4\,\omega\,\omega_{-B}\,\omega_{-a,\,b} + \frac{1}{2}\,\mu_{-B}\,\omega_{-a,\,b} + 8\,\omega\,\theta\,P2_{-a,\,b}\,\omega_{-B} - \mu\,P_{-a,\,b}\,\theta_{-B} + \frac{16}{3}\,\omega^2\,P2_{-a,\,b}\,\theta_{-B} + \left(\omega_{-a,\,b}\,\theta - \frac{1}{2}\,\omega_{-a,\,c}\,\omega_{-c,\,b}\right)\theta_{-B} + \left(\frac{1}{3}\,\theta\,\mu_{-B} - 8\,\omega\,\omega_{-B}\,\omega_{-a,\,b} - \frac{40}{3}\,\omega\,\theta\,\omega_{-B}\right)P_{-a,\,b} = 0 \quad \textbf{(1.1)}$$

```
> eq[43] := (1/2)·(mu + p - (16/3)·ω^2)·omega[-a, b]·theta[-B] + theta·((112/9)
      ·omega·omega - (5/3)·(mu + p))·P[-a, b]·theta[-B] + ((5/9)·theta·theta - (2/3)
```

·omega·omega + $\left(\frac{1}{6}\right)$·(mu + 3·*p*))*P*[−*a*, *b*]·mu[−*B*] + $\left(\frac{7}{6}\right)$·theta·omega[−*a*, *c*]·omega[−*c*, *b*]·theta[−*B*] − $\left(\frac{1}{4}\right)$·omega[−*a*, *d*]·omega[−*d*, *c*]·omega[−*c*, *b*]·theta[−*B*] = 0 : *T*(%);

$$\frac{1}{2}\left(\mu+p-\frac{16}{3}\omega^2\right)\omega_a{}^b\theta_{;b}+\theta\left(\frac{112}{9}\omega^2-\frac{5}{3}\mu-\frac{5}{3}p\right)P_a{}^b\theta_{;b}+\left(\frac{5}{9}\theta^2-\frac{2}{3}\omega^2+\frac{1}{6}\mu+\frac{1}{2}p\right)P_a{}^b\mu_{;b}+\frac{7}{6}\theta\,\omega_a{}^c\omega_c{}^b\theta_{;b}-\frac{1}{4}\omega_a{}^d\omega_d{}^c\omega_c{}^b\theta_{;b}=0 \quad \textbf{(1.2)}$$

****

proof of eq43:

We commence with eq42 :

> *eq*[42] := (mu − $\left(\frac{16}{3}\right)$·omega·omega)·*P*[−*a*, *b*]·theta[−*B*] = $\left(\frac{1}{2}\right)$·omega[−*a*, *c*]·omega[−*c*, *b*]·theta[−*B*] + $\left(\frac{\text{theta}}{3}\right)$·*P*[−*a*, *b*]·mu[−*B*] : *T*(%);

$$\left(\mu-\frac{16}{3}\omega^2\right)P_a{}^b\theta_{;b}=\frac{1}{2}\omega_a{}^c\omega_c{}^b\theta_{;b}+\frac{1}{3}\theta P_a{}^b\mu_{;b} \quad \textbf{(1.3)}$$

step 1. substituting eq40:

> *temp* := *expand*$\left(\frac{1}{3}\right.$·theta·*eq*[40]$\left.\right)$: *T*(%);

$$-\frac{8}{3}\omega\theta P_a{}^b\omega_{;b}+\frac{1}{3}\theta P_a{}^b\mu_{;b}+\frac{1}{3}\theta\omega_a{}^b\theta_{;b}=0 \quad \textbf{(1.4)}$$

> #temp2:=op(1, *op*(1, temp)) =−op(2, op(1, temp)) − op(3, op(1, temp)) : *T*(%);

> *temp2* := *isolate*(*temp*, −$\frac{8}{3}$·theta·omega·*P*[−*a*, *b*]·omega[−*B*]) : *T*(%)

$$-\frac{8}{3}\omega\theta P_a{}^b\omega_{;b}=-\frac{1}{3}\theta P_a{}^b\mu_{;b}-\frac{1}{3}\theta\omega_a{}^b\theta_{;b} \quad \textbf{(1.5)}$$

> *temp3* := 3· *TEDS*(*temp2*, *eq*[42]) : *T*(%);

$$3\left(\mu-\frac{16}{3}\omega^2\right)P_a{}^b\theta_{;b}=\frac{3}{2}\omega_a{}^c\omega_c{}^b\theta_{;b}+\theta P_a{}^b\mu_{;b} \quad \textbf{(1.6)}$$

> *eq43a* := *temp3*;

$$eq43a := 3\left(\mu-\frac{16}{3}\omega^2\right)P_{-a,b}\theta_{-B}=\frac{3}{2}\omega_{-a,c}\omega_{-c,b}\theta_{-B}+\theta P_{-a,b}\mu_{-B} \quad \textbf{(1.7)}$$

> **save** *eq43a*, "eq43a";

2. multiplying by $\omega_e{}^a$

> *temp4* := *expand*(omega[−*e*, *a*]·*temp3*) : *T*(%);

$$-16\,\omega_e{}^aP_a{}^b\theta_{;b}\omega^2+3\,\omega_e{}^aP_a{}^b\theta_{;b}\mu=\frac{3}{2}\omega_e{}^a\omega_a{}^c\omega_c{}^b\theta_{;b}+\omega_e{}^a\theta P_a{}^b\mu_{;b} \quad \textbf{(1.8)}$$

Now subs the identity for P

> temp := P[-a, b]=g[-a, b] + u[-a]·u[b] : T(%);

$$P_a{}^b = u_a u^b + g_a{}^b \quad (1.9)$$

> temp3 := TEDS(temp, temp4) : T(%);

$$-16\,\omega^2 \omega_e{}^a \theta_{;b} u^b u_a + 3\,\mu\,\omega_e{}^a \theta_{;b} u^b u_a - 16\,\omega^2 g_a{}^b \omega_e{}^a \theta_{;b} + 3\,\mu\, g_a{}^b \omega_e{}^a \theta_{;b} = \frac{3}{2}\,\omega_e{}^a \omega_a{}^c \omega_c{}^b \theta_{;b} + \theta\,\mu_{;b}\,\omega_e{}^a u^b u_a + \theta\, g_a{}^b \mu_{;b}\,\omega_e{}^a \quad (1.10)$$

> temp4 := Absorbg(temp3) : T(%);

$$-16\,\omega^2 \omega_e{}^a \theta_{;b} u^b u_a + 3\,\mu\,\omega_e{}^a \theta_{;b} u^b u_a - 16\,\omega_e{}^b \theta_{;b}\,\omega^2 + 3\,\omega_e{}^b \theta_{;b}\,\mu = \frac{3}{2}\,\omega_e{}^a \omega_a{}^c \omega_c{}^b \theta_{;b} + \theta\,\mu_{;b}\,\omega_e{}^a u^b u_a + \omega_e{}^b \theta\,\mu_{;b} \quad (1.11)$$

> temp5 := $\frac{1}{3}$·TEDS(omega[-e, a]·u[-a]=0, temp4) : T(%);

$$-\frac{16}{3}\,\omega_e{}^b \theta_{;b}\,\omega^2 + \omega_e{}^b \theta_{;b}\,\mu = \frac{1}{2}\,\omega_e{}^a \omega_a{}^c \omega_c{}^b \theta_{;b} + \frac{1}{3}\,\omega_e{}^b \theta\,\mu_{;b} \quad (1.12)$$

> temp6 := collect(temp5, [omega[-e, b], theta[-B]], `distributed`) : T(%);

$$\left(\mu - \frac{16}{3}\,\omega^2\right)\theta_{;b}\,\omega_e{}^b = \frac{1}{2}\,\omega_e{}^a \omega_a{}^c \omega_c{}^b \theta_{;b} + \frac{1}{3}\,\omega_e{}^b \theta\,\mu_{;b} \quad (1.13)$$

> temp7 := subs(a=d, e=a, lhs(temp6) = expand(rhs(temp6))) : T(%);

$$\left(\mu - \frac{16}{3}\,\omega^2\right)\theta_{;b}\,\omega_a{}^b = \frac{1}{2}\,\omega_a{}^d \omega_d{}^c \omega_c{}^b \theta_{;b} + \frac{1}{3}\,\omega_a{}^b \theta\,\mu_{;b} \quad (1.14)$$

now subs θ*eq41

> temp := expand(theta·eq[41]) : T(%);

$$-8\,\omega\,\theta\,\omega_{;b}\,\omega_a{}^b + \theta\,\omega_a{}^c \omega_c{}^b \theta_{;b} + \omega_a{}^b \theta\,\mu_{;b} = 0 \quad (1.15)$$

> #temp2:=op(1, op(1, temp)) =-op(2, op(1, temp)) − op(3, op(1, temp)) : T(%);

> temp2 := isolate(temp, -8·omega[-a, b]·theta·omega·omega[-B]) : T(%);

$$-8\,\omega\,\theta\,\omega_{;b}\,\omega_a{}^b = -\theta\,\omega_a{}^c \omega_c{}^b \theta_{;b} - \omega_a{}^b \theta\,\mu_{;b} \quad (1.16)$$

> temp3 := -temp2 : T(%);

$$8\,\omega\,\theta\,\omega_{;b}\,\omega_a{}^b = \theta\,\omega_a{}^c \omega_c{}^b \theta_{;b} + \omega_a{}^b \theta\,\mu_{;b} \quad (1.17)$$

> temp8 := $\frac{1}{3}$·TEDS(temp3, temp7) : T(%);

$$\frac{1}{3}\left(\mu - \frac{16}{3}\,\omega^2\right)\theta_{;b}\,\omega_a{}^b = \frac{1}{6}\,\omega_a{}^d \omega_d{}^c \omega_c{}^b \theta_{;b} + \frac{1}{9}\,\omega_a{}^b \theta\,\mu_{;b} \quad (1.18)$$

> temp9 := TEDS(omega[-a, d]·omega[-d, b]=omega[-a, c]·omega[-c, b], temp8) : T(%);

$$\frac{1}{3}\left(\mu - \frac{16}{3}\,\omega^2\right)\theta_{;b}\,\omega_a{}^b = \frac{1}{6}\,\omega_a{}^d \omega_d{}^c \omega_c{}^b \theta_{;b} + \frac{1}{9}\,\omega_a{}^b \theta\,\mu_{;b} \quad (1.19)$$

```
> step42b := 3· temp9 : T(%);
```

$$\left(\mu - \frac{16}{3}\,\omega^2\right)\theta_{;b}\,\omega_a{}^b = \frac{1}{2}\,\omega_a{}^d\,\omega_d{}^c\,\omega_c{}^b\,\theta_{;b} + \frac{1}{3}\,\omega_a{}^b\,\theta\,\mu_{;b} \qquad \textbf{(1.20)}$$

3: using eq14b: $\omega_a{}^c\,\omega_c{}^d\,\omega_d{}^b = -\omega^2\,\omega_a{}^b$

```
> temp := subs(c = e, d = c, e = d, eq[14 b]) : T(%);
```

$$\omega_a{}^d\,\omega_d{}^c\,\omega_c{}^b = -\omega^2\,\omega_a{}^b \qquad \textbf{(1.21)}$$

```
> temp10 := TEDS(temp, temp9) : T(%);
```

$$\frac{1}{3}\left(\mu - \frac{16}{3}\,\omega^2\right)\theta_{;b}\,\omega_a{}^b = -\frac{1}{6}\,\omega^2\,\omega_a{}^b\,\theta_{;b} + \frac{1}{9}\,\omega_a{}^b\,\theta\,\mu_{;b} \qquad \textbf{(1.22)}$$

4: we set up 2 equations:

```
> eq1 := expand(omega[a]·eq[40]) : T(%);
```

$$-8\,\omega\,P_a{}^b\,\omega^a\,\omega_{;b} + P_a{}^b\,\mu_{;b}\,\omega^a + \omega^a\,\omega_a{}^b\,\theta_{;b} = 0 \qquad \textbf{(1.23)}$$

```
> eq1 := expand(TEDS(omega[a]·omega[-a, b] = 0, eq1)) : T(%);
```

$$-8\,\omega\,P_a{}^b\,\omega^a\,\omega_{;b} + P_a{}^b\,\mu_{;b}\,\omega^a = 0 \qquad \textbf{(1.24)}$$

```
> eq1 := expand(TEDS(omega[a]·P[-a, b] = omega[b], eq1)) : T(%);
```

$$-8\,\omega\,\omega_{;b}\,\omega^b + \mu_{;b}\,\omega^b = 0 \qquad \textbf{(1.25)}$$

```
> eq2 := expand(omega[a]·eq[42]) : T(%);
```

$$\omega^a\,P_a{}^b\,\theta_{;b}\,\mu - \frac{16}{3}\,\omega^a\,P_a{}^b\,\theta_{;b}\,\omega^2 = \frac{1}{2}\,\omega^a\,\omega_a{}^c\,\omega_c{}^b\,\theta_{;b} + \frac{1}{3}\,\omega^a\,\theta\,P_a{}^b\,\mu_{;b} \qquad \textbf{(1.26)}$$

```
> eq2 := expand(TEDS(omega[a]·omega[-a, c] = 0, eq2)) : T(%);
```

$$\omega^a\,P_a{}^b\,\theta_{;b}\,\mu - \frac{16}{3}\,\omega^a\,P_a{}^b\,\theta_{;b}\,\omega^2 = \frac{1}{3}\,\omega^a\,\theta\,P_a{}^b\,\mu_{;b} \qquad \textbf{(1.27)}$$

```
> eq2 := expand(TEDS(omega[a]·P[-a, b] = omega[b], eq2)) : T(%);
```

$$\theta_{;b}\,\mu\,\omega^b - \frac{16}{3}\,\theta_{;b}\,\omega^2\,\omega^b = \frac{1}{3}\,\theta\,\mu_{;b}\,\omega^b \qquad \textbf{(1.28)}$$

```
> eq1 := eq1 − op(1, op(1, eq1)) : T(%);
```

$$\mu_{;b}\,\omega^b = 8\,\omega\,\omega_{;b}\,\omega^b \qquad \textbf{(1.29)}$$

```
> eq12 := 3· expand(TEDS(eq1, eq2)) : T(%);
```

$$-16\,\theta_{;b}\,\omega^2\,\omega^b + 3\,\theta_{;b}\,\mu\,\omega^b = 8\,\theta\,\omega\,\omega^b\,\omega_{;b} \qquad \textbf{(1.30)}$$

5:eq42 with id

```
> step5a := expand(TEDS(eq[14 a], eq[42])) : T(%);
```

$$P_a{}^b\,\theta_{;b}\,\mu - \frac{16}{3}\,P_a{}^b\,\theta_{;b}\,\omega^2 = -\frac{1}{2}\,P_a{}^b\,\theta_{;b}\,\omega^2 + \frac{1}{2}\,\omega^b\,\omega_a\,\theta_{;b} + \frac{1}{3}\,\theta\,P_a{}^b\,\mu_{;b} \qquad \textbf{(1.31)}$$

> $step5b := step5a + \frac{1}{2} \cdot P[-a, b] \cdot \text{theta}[-B] \cdot \text{omega} \cdot \text{omega} : T(\%);$

$$P_a{}^b \theta_{;b} \mu - \frac{29}{6} P_a{}^b \theta_{;b} \omega^2 = \frac{1}{2} \omega^b \omega_a \theta_{;b} + \frac{1}{3} \theta P_a{}^b \mu_{;b} \quad \textbf{(1.32)}$$

6:step6
commencing by time propogating eq43

> $eq[42] : T(\%);$

$$\left(\mu - \frac{16}{3} \omega^2\right) P_a{}^b \theta_{;b} = \frac{1}{2} \omega_a{}^c \omega_c{}^b \theta_{;b} + \frac{1}{3} \theta P_a{}^b \mu_{;b} \quad \textbf{(1.33)}$$

> $step6 := dotT(eq[42]) : T(\%);$

$$\left(dotmu - \frac{32}{3} \omega\, dotomega\right) P_a{}^b \theta_{;b} + \left(\mu - \frac{16}{3} \omega^2\right) dotP_a{}^b \theta_{;b} + \left(\mu - \frac{16}{3} \omega^2\right) P_a{}^b dottheta_{;b} = \frac{1}{2} dotomega_a{}^c \omega_c{}^b \theta_{;b} + \frac{1}{2} \omega_a{}^c dotomega_c{}^b \theta_{;b} + \frac{1}{2} \omega_a{}^c \omega_c{}^b dottheta_{;b} + \frac{1}{3} dottheta\, P_a{}^b \mu_{;b} + \frac{1}{3} \theta\, dotP_a{}^b \mu_{;b} + \frac{1}{3} \theta P_a{}^b dotmu_{;b} \quad \textbf{(1.34)}$$

> $step6a := TEDS(dotmu = -\text{mu} \cdot \text{theta}, step6) : T(\%);$

$$P_a{}^b \theta_{;b} \left(-\theta \mu - \frac{32}{3} \omega\, dotomega\right) + \left(\mu - \frac{16}{3} \omega^2\right) dotP_a{}^b \theta_{;b} + \left(\mu - \frac{16}{3} \omega^2\right) P_a{}^b dottheta_{;b} = \frac{1}{2} dotomega_a{}^c \omega_c{}^b \theta_{;b} + \frac{1}{2} \omega_a{}^c dotomega_c{}^b \theta_{;b} + \frac{1}{2} \omega_a{}^c \omega_c{}^b dottheta_{;b} + \frac{1}{3} dottheta\, P_a{}^b \mu_{;b} + \frac{1}{3} \theta\, dotP_a{}^b \mu_{;b} + \frac{1}{3} \theta P_a{}^b dotmu_{;b} \quad \textbf{(1.35)}$$

> $step6b := expand\left(TEDS\left(dotomega = -\frac{2}{3} \cdot \text{omega} \cdot \text{theta}, step6a\right)\right) : T(\%);$

$$-P_a{}^b \theta_{;b} \theta \mu + \frac{64}{9} P_a{}^b \theta_{;b} \omega^2 \theta + dotP_a{}^b \theta_{;b} \mu - \frac{16}{3} dotP_a{}^b \theta_{;b} \omega^2 + P_a{}^b dottheta_{;b} \mu - \frac{16}{3} P_a{}^b dottheta_{;b} \omega^2 = \frac{1}{2} dotomega_a{}^c \omega_c{}^b \theta_{;b} + \frac{1}{2} \omega_a{}^c dotomega_c{}^b \theta_{;b} + \frac{1}{2} \omega_a{}^c \omega_c{}^b dottheta_{;b} + \frac{1}{3} dottheta\, P_a{}^b \mu_{;b} + \frac{1}{3} \theta\, dotP_a{}^b \mu_{;b} + \frac{1}{3} \theta P_a{}^b dotmu_{;b} \quad \textbf{(1.36)}$$

> $step6c := expand(TEDS(dotP[-a, b] = 0, step6b)) : T(\%);$

$$-P_a{}^b \theta_{;b} \theta \mu + \frac{64}{9} P_a{}^b \theta_{;b} \omega^2 \theta + P_a{}^b dottheta_{;b} \mu - \frac{16}{3} P_a{}^b dottheta_{;b} \omega^2 \quad \textbf{(1.37)}$$

$$= \frac{1}{2}\, dotomega_{a}{}^{c}\, \omega_{c}{}^{b}\, \theta_{;b} + \frac{1}{2}\, \omega_{a}{}^{c}\, dotomega_{c}{}^{b}\, \theta_{;b} + \frac{1}{2}\, \omega_{a}{}^{c}\, \omega_{c}{}^{b}\, dottheta_{;b} + \frac{1}{3}\, dottheta\, P_{a}{}^{b}\, \mu_{;b} + \frac{1}{3}\, \theta\, P_{a}{}^{b}\, dotmu_{;b}$$

> temp := isolate(eq[20], dottheta) : T(%);

$$dottheta = -\frac{1}{3}\, \theta^{2} + 2\, \omega^{2} - \frac{1}{2}\, \mu \tag{1.38}$$

> tdot := expand(TEDS(dottheta = thetadot, temp)) : T(%);

$$thetadot = -\frac{1}{3}\, \theta^{2} + 2\, \omega^{2} - \frac{1}{2}\, \mu \tag{1.39}$$

> temp3 := cod(tdot, -b) : T(%);

$$thetadot_{;b} = -\frac{2}{3}\, \theta\, \theta_{;b} + 4\, \omega\, \omega_{;b} - \frac{1}{2}\, \mu_{;b} \tag{1.40}$$

and using eq38 (rewriting)

> temp := P[-a, b]·dottheta[-B] = P[-a, b]·thetadot[-B] + omega[-a, b]·theta[-B] − $\frac{1}{3}$ ·theta·P[-a, b]·theta[-B] : T(%);

$$P_{a}{}^{b}\, dottheta_{;b} = P_{a}{}^{b}\, thetadot_{;b} + \omega_{a}{}^{b}\, \theta_{;b} - \frac{1}{3}\, \theta\, P_{a}{}^{b}\, \theta_{;b} \tag{1.41}$$

> step6d := expand(TEDS(temp, step6c)) : T(%);

$$-\frac{4}{3}\, P_{a}{}^{b}\, \theta_{;b}\, \theta\, \mu + \frac{80}{9}\, P_{a}{}^{b}\, \theta_{;b}\, \omega^{2}\, \theta + P_{a}{}^{b}\, \mu\, thetadot_{;b} - \frac{16}{3}\, P_{a}{}^{b}\, \omega^{2}\, thetadot_{;b} + \mu\, \omega_{a}{}^{b}\, \theta_{;b} - \frac{16}{3}\, \omega^{2}\, \omega_{a}{}^{b}\, \theta_{;b} = \frac{1}{2}\, dotomega_{a}{}^{c}\, \omega_{c}{}^{b}\, \theta_{;b} + \frac{1}{2}\, \omega_{a}{}^{c}\, dotomega_{c}{}^{b}\, \theta_{;b} + \frac{1}{2}\, \omega_{a}{}^{c}\, \omega_{c}{}^{b}\, dottheta_{;b} + \frac{1}{3}\, dottheta\, P_{a}{}^{b}\, \mu_{;b} + \frac{1}{3}\, \theta\, P_{a}{}^{b}\, dotmu_{;b} \tag{1.42}$$

> T(temp2);

$$-8\, \omega\, \theta\, \omega_{;b}\, \omega_{a}{}^{b} = -\theta\, \omega_{a}{}^{c}\, \omega_{c}{}^{b}\, \theta_{;b} - \omega_{a}{}^{b}\, \theta\, \mu_{;b} \tag{1.43}$$

> step6e := expand(TEDS(temp2, step6d)) : T(%);

$$-\frac{4}{3}\, P_{a}{}^{b}\, \theta_{;b}\, \theta\, \mu + \frac{80}{9}\, P_{a}{}^{b}\, \theta_{;b}\, \omega^{2}\, \theta + P_{a}{}^{b}\, \mu\, thetadot_{;b} - \frac{16}{3}\, P_{a}{}^{b}\, \omega^{2}\, thetadot_{;b} + \mu\, \omega_{a}{}^{b}\, \theta_{;b} - \frac{16}{3}\, \omega^{2}\, \omega_{a}{}^{b}\, \theta_{;b} = \frac{1}{2}\, dotomega_{a}{}^{c}\, \omega_{c}{}^{b}\, \theta_{;b} + \frac{1}{2}\, \omega_{a}{}^{c}\, dotomega_{c}{}^{b}\, \theta_{;b} + \frac{1}{2}\, \omega_{a}{}^{c}\, \omega_{c}{}^{b}\, dottheta_{;b} + \frac{1}{3}\, dottheta\, P_{a}{}^{b}\, \mu_{;b} + \frac{1}{3}\, \theta\, P_{a}{}^{b}\, dotmu_{;b} \tag{1.44}$$

> step6f := expand(TEDS(temp3, step6e)) : T(%);

$$-2 P_a{}^b \theta_{;b} \theta \mu + \frac{112}{9} P_a{}^b \theta_{;b} \omega^2 \theta + 4 P_a{}^b \mu \omega \omega_{;b} - \frac{1}{2} P_a{}^b \mu \mu_{;b} - \frac{64}{3} P_a{}^b \omega^3 \omega_{;b} + \frac{8}{3} P_a{}^b \omega^2 \mu_{;b} + \mu \omega_a{}^b \theta_{;b} - \frac{16}{3} \omega^2 \omega_a{}^b \theta_{;b} = \frac{1}{2} dotomega_a{}^c \omega_c{}^b \theta_{;b} + \frac{1}{2} \omega_a{}^c dotomega_c{}^b \theta_{;b} + \frac{1}{2} \omega_a{}^c \omega_c{}^b dottheta_{;b} + \frac{1}{3} dottheta P_a{}^b \mu_{;b} + \frac{1}{3} \theta P_a{}^b dotmu_{;b} \tag{1.45}$$

> step6g := expand(TEDS(dotomega[-a, c] =- $\frac{2}{3}$ ·theta·omega[-a, c], step6f)) : T(%);

$$-2 P_a{}^b \theta_{;b} \theta \mu + \frac{112}{9} P_a{}^b \theta_{;b} \omega^2 \theta + 4 P_a{}^b \mu \omega \omega_{;b} - \frac{1}{2} P_a{}^b \mu \mu_{;b} - \frac{64}{3} P_a{}^b \omega^3 \omega_{;b} + \frac{8}{3} P_a{}^b \omega^2 \mu_{;b} + \mu \omega_a{}^b \theta_{;b} - \frac{16}{3} \omega^2 \omega_a{}^b \theta_{;b} = -\frac{1}{3} \theta \omega_a{}^c \omega_c{}^b \theta_{;b} + \frac{1}{2} \omega_a{}^c dotomega_c{}^b \theta_{;b} + \frac{1}{2} \omega_a{}^c \omega_c{}^b dottheta_{;b} + \frac{1}{3} dottheta P_a{}^b \mu_{;b} + \frac{1}{3} \theta P_a{}^b dotmu_{;b} \tag{1.46}$$

Now we need an identity for $dottheta_{;b}$

> temp := dottheta[-B] = theta[-B, -E]·u[e] : T(%);

$$dottheta_{;b} = \theta_{;b;e} u^e \tag{1.47}$$

which, because of torsion-free assumption of scalars, can be written, in reverse

> temp := theta[-E, -B]·u[e] = dottheta[-B] : T(%);

$$\theta_{;e;b} u^e = dottheta_{;b} \tag{1.48}$$

Now, using eq18

> temp1 := thetadot[-B] = cod(theta[-E]·u[e], -b) : T(%);

$$thetadot_{;b} = \theta_{;e} u^e{}_{;b} + \theta_{;e;b} u^e \tag{1.49}$$

> temp2 := TEDS(temp, temp1) : T(%);

$$thetadot_{;b} = \theta_{;e} u^e{}_{;b} + dottheta_{;b} \tag{1.50}$$

> temp3 := isolate(temp2, dottheta[-B]) : T(%);

$$dottheta_{;b} = -\theta_{;e} u^e{}_{;b} + thetadot_{;b} \tag{1.51}$$

using the correct version of eq18

> eq[18] := u[-a, -B] = $\frac{1}{3}$ ·theta·P[-a, -b] + omega[-a, -b] : T(%);

$$u_{a;b} = \frac{1}{3} \theta P_{ab} + \omega_{ab} \tag{1.52}$$

> temp4 := TEDS(subs(a=-e, A=-E, eq[18]), temp3) : T(%);

$$dottheta_{;b} = -\frac{1}{3}\theta_{;e}\theta P^{e}{}_{b} - \theta_{;e}\omega^{e}{}_{b} + thetadot_{;b} \quad (1.53)$$

> temp4a := TEDS(P[e,-b]=P[-b, e], temp4) : T(%);

$$dottheta_{;b} = -\frac{1}{3}\theta_{;e}\theta P_{b}{}^{e} - \theta_{;e}\omega^{e}{}_{b} + thetadot_{;b} \quad (1.54)$$

> temp5 := cod(tdot,-b) : T(%);

$$thetadot_{;b} = -\frac{2}{3}\theta\theta_{;b} + 4\omega\omega_{;b} - \frac{1}{2}\mu_{;b} \quad (1.55)$$

> temp6 := TEDS(temp5, temp4a) : T(%);

$$dottheta_{;b} = -\frac{1}{3}\theta_{;e}\theta P_{b}{}^{e} - \theta_{;e}\omega^{e}{}_{b} - \frac{2}{3}\theta\theta_{;b} + 4\omega\omega_{;b} - \frac{1}{2}\mu_{;b} \quad (1.56)$$

> temp7 := Absorbg(TEDS(P[-b, e]=g[-b, e]+u[-b]·u[e], temp6)) : T(%);

$$dottheta_{;b} = -\frac{1}{3}\theta\theta_{;e}u^{e}u_{b} - \theta\theta_{;b} - \theta_{;e}\omega^{e}{}_{b} + 4\omega\omega_{;b} - \frac{1}{2}\mu_{;b} \quad (1.57)$$

> temp7a := TEDS(P[-b, e]=g[-b, e]+u[-b]·u[e], temp6) : T(%);

$$dottheta_{;b} = -\frac{1}{3}\theta\theta_{;e}u^{e}u_{b} - \frac{1}{3}\theta g_{b}{}^{e}\theta_{;e} - \theta_{;e}\omega^{e}{}_{b} - \frac{2}{3}\theta\theta_{;b} + 4\omega\omega_{;b} - \frac{1}{2}\mu_{;b} \quad (1.58)$$

> temp7 := Absorbg(temp7a) : T(%);

$$dottheta_{;b} = -\frac{1}{3}\theta\theta_{;e}u^{e}u_{b} - \theta\theta_{;b} - \theta_{;e}\omega^{e}{}_{b} + 4\omega\omega_{;b} - \frac{1}{2}\mu_{;b} \quad (1.59)$$

>

replacing in the main term we have:

> step6h := TEDS(temp7, step6g) : T(%);

$$\begin{aligned}&-2P_{a}{}^{b}\theta_{;b}\theta\mu + \frac{112}{9}P_{a}{}^{b}\theta_{;b}\omega^{2}\theta + 4P_{a}{}^{b}\mu\omega\omega_{;b} - \frac{1}{2}P_{a}{}^{b}\mu\mu_{;b} \\ &- \frac{64}{3}P_{a}{}^{b}\omega^{3}\omega_{;b} + \frac{8}{3}P_{a}{}^{b}\omega^{2}\mu_{;b} + \mu\omega_{a}{}^{b}\theta_{;b} - \frac{16}{3}\omega^{2}\omega_{a}{}^{b}\theta_{;b} = \\ &-\frac{5}{6}\theta\omega_{a}{}^{c}\omega_{c}{}^{b}\theta_{;b} + \frac{1}{2}\omega_{a}{}^{c}dotomega_{c}{}^{b}\theta_{;b} - \frac{1}{6}\omega_{a}{}^{c}\omega_{c}{}^{b}\theta\theta_{;e}u^{e}u_{b} \\ &- \frac{1}{2}\omega_{a}{}^{c}\omega_{c}{}^{b}\theta_{;e}\omega^{e}{}_{b} + 2\omega_{a}{}^{c}\omega_{c}{}^{b}\omega\omega_{;b} - \frac{1}{4}\omega_{a}{}^{c}\omega_{c}{}^{b}\mu_{;b} \\ &+ \frac{1}{3}dottheta P_{a}{}^{b}\mu_{;b} + \frac{1}{3}\theta P_{a}{}^{b}dotmu_{;b}\end{aligned} \quad (1.60)$$

> step6i := expand(TEDS(omega[-c, b]·u[-b]=0, step6h)) : T(%);

$$\begin{aligned}&-2P_{a}{}^{b}\theta_{;b}\theta\mu + \frac{112}{9}P_{a}{}^{b}\theta_{;b}\omega^{2}\theta + 4P_{a}{}^{b}\mu\omega\omega_{;b} - \frac{1}{2}P_{a}{}^{b}\mu\mu_{;b} \\ &- \frac{64}{3}P_{a}{}^{b}\omega^{3}\omega_{;b} + \frac{8}{3}P_{a}{}^{b}\omega^{2}\mu_{;b} + \mu\omega_{a}{}^{b}\theta_{;b} - \frac{16}{3}\omega^{2}\omega_{a}{}^{b}\theta_{;b} =\end{aligned} \quad (1.61)$$

$$-\frac{5}{6}\,\theta\,\omega_a{}^c\,\omega_c{}^b\,\theta_{;b}+\frac{1}{2}\,\omega_a{}^c\,dotomega_c{}^b\,\theta_{;b}-\frac{1}{2}\,\omega_a{}^c\,\omega_c{}^b\,\theta_{;e}\,\omega^e{}_b$$
$$+2\,\omega_a{}^c\,\omega_c{}^b\,\omega\,\omega_{;b}-\frac{1}{4}\,\omega_a{}^c\,\omega_c{}^b\,\mu_{;b}+\frac{1}{3}\,dottheta\,P_a{}^b\,\mu_{;b}$$
$$+\frac{1}{3}\,\theta\,P_a{}^b\,dotmu_{;b}$$

and using eq38 (rewriting for mu)

```
> temp := P[-a, b]·dotmu[-B] = P[-a, b]·mudot[-B] + omega[-a, b]·mu[-B] - 1/3·theta
    ·P[-a, b]·mu[-B] : T(%);
```

$$P_a{}^b\,dotmu_{;b}=P_a{}^b\,mudot_{;b}+\omega_a{}^b\,\mu_{;b}-\frac{1}{3}\,\theta\,P_a{}^b\,\mu_{;b}\qquad\textbf{(1.62)}$$

```
> mudot[-B] = cod(-mu·theta, -b) : T(%);
```

$$mudot_{;b}=-\mu\,\theta_{;b}-\mu\,\theta_{;b}\qquad\textbf{(1.63)}$$

```
> step6j1 := TEDS(mudot[-B] = cod(-mu·theta, -b), temp) : T(%);
```

$$P_a{}^b\,dotmu_{;b}=-P_a{}^b\,\theta_{;b}\,\mu-\frac{4}{3}\,\theta\,P_a{}^b\,\mu_{;b}+\omega_a{}^b\,\mu_{;b}\qquad\textbf{(1.64)}$$

```
>
```

```
> step6j := expand(TEDS(step6j1, step6i)) : T(%);
```

$$-2\,P_a{}^b\,\theta_{;b}\,\theta\,\mu+\frac{112}{9}\,P_a{}^b\,\theta_{;b}\,\omega^2\,\theta+4\,P_a{}^b\,\mu\,\omega\,\omega_{;b}-\frac{1}{2}\,P_a{}^b\,\mu\,\mu_{;b}\qquad\textbf{(1.65)}$$
$$-\frac{64}{3}\,P_a{}^b\,\omega^3\,\omega_{;b}+\frac{8}{3}\,P_a{}^b\,\omega^2\,\mu_{;b}+\mu\,\omega_a{}^b\,\theta_{;b}-\frac{16}{3}\,\omega^2\,\omega_a{}^b\,\theta_{;b}$$
$$=\frac{1}{3}\,dottheta\,P_a{}^b\,\mu_{;b}-\frac{1}{3}\,P_a{}^b\,\theta_{;b}\,\theta\,\mu-\frac{4}{9}\,P_a{}^b\,\theta^2\,\mu_{;b}-\frac{5}{6}\,\theta\,\omega_a{}^c\,\omega_c{}^b\,\theta_{;b}$$
$$+\frac{1}{2}\,\omega_a{}^c\,dotomega_c{}^b\,\theta_{;b}-\frac{1}{2}\,\omega_a{}^c\,\omega_c{}^b\,\theta_{;e}\,\omega^e{}_b+2\,\omega_a{}^c\,\omega_c{}^b\,\omega\,\omega_{;b}$$
$$-\frac{1}{4}\,\omega_a{}^c\,\omega_c{}^b\,\mu_{;b}+\frac{1}{3}\,\omega_a{}^b\,\theta\,\mu_{;b}$$

```
> temp := isolate(eq[20], dottheta) : T(%);
```

$$dottheta=-\frac{1}{3}\,\theta^2+2\,\omega^2-\frac{1}{2}\,\mu\qquad\textbf{(1.66)}$$

```
> step6k := expand(TEDS(temp, step6j)) : T(%);
```

$$-2\,P_a{}^b\,\theta_{;b}\,\theta\,\mu+\frac{112}{9}\,P_a{}^b\,\theta_{;b}\,\omega^2\,\theta+4\,P_a{}^b\,\mu\,\omega\,\omega_{;b}-\frac{1}{2}\,P_a{}^b\,\mu\,\mu_{;b}\qquad\textbf{(1.67)}$$
$$-\frac{64}{3}\,P_a{}^b\,\omega^3\,\omega_{;b}+\frac{8}{3}\,P_a{}^b\,\omega^2\,\mu_{;b}+\mu\,\omega_a{}^b\,\theta_{;b}-\frac{16}{3}\,\omega^2\,\omega_a{}^b\,\theta_{;b}=$$
$$-\frac{5}{9}\,P_a{}^b\,\theta^2\,\mu_{;b}+\frac{2}{3}\,P_a{}^b\,\omega^2\,\mu_{;b}-\frac{1}{6}\,P_a{}^b\,\mu\,\mu_{;b}-\frac{1}{3}\,P_a{}^b\,\theta_{;b}\,\theta\,\mu$$

$$-\frac{5}{6}\theta\omega_a{}^c\omega_c{}^b\theta_{;b}+\frac{1}{2}\omega_a{}^c dotomega_c{}^b\theta_{;b}-\frac{1}{2}\omega_a{}^c\omega_c{}^b\theta_{;e}\omega^e{}_b+2\omega_a{}^c\omega_c{}^b\omega\omega_{;b}-\frac{1}{4}\omega_a{}^c\omega_c{}^b\mu_{;b}+\frac{1}{3}\omega_a{}^b\theta\mu_{;b}$$

> *step6l* := *expand*(*TEDS*(*dotomega*[-*c*, *b*] =- $\frac{2}{3}$ ·omega[-*c*, *b*]·theta, *step6k*)) : *T*(%);

$$-2P_a{}^b\theta_{;b}\theta\mu+\frac{112}{9}P_a{}^b\theta_{;b}\omega^2\theta+4P_a{}^b\mu\omega\omega_{;b}-\frac{1}{2}P_a{}^b\mu\mu_{;b}-\frac{64}{3}P_a{}^b\omega^3\omega_{;b}+\frac{8}{3}P_a{}^b\omega^2\mu_{;b}+\mu\omega_a{}^b\theta_{;b}-\frac{16}{3}\omega^2\omega_a{}^b\theta_{;b}= -\frac{5}{9}P_a{}^b\theta^2\mu_{;b}+\frac{2}{3}P_a{}^b\omega^2\mu_{;b}-\frac{1}{6}P_a{}^b\mu\mu_{;b}-\frac{1}{3}P_a{}^b\theta_{;b}\theta\mu-\frac{7}{6}\theta\omega_a{}^c\omega_c{}^b\theta_{;b}-\frac{1}{2}\omega_a{}^c\omega_c{}^b\theta_{;e}\omega^e{}_b+2\omega_a{}^c\omega_c{}^b\omega\omega_{;b}-\frac{1}{4}\omega_a{}^c\omega_c{}^b\mu_{;b}+\frac{1}{3}\omega_a{}^b\theta\mu_{;b} \qquad \textbf{(1.68)}$$

> *TERM43l* := *lhs*(*step6l*) : *T*(%);

$$-2P_a{}^b\theta_{;b}\theta\mu+\frac{112}{9}P_a{}^b\theta_{;b}\omega^2\theta+4P_a{}^b\mu\omega\omega_{;b}-\frac{1}{2}P_a{}^b\mu\mu_{;b}-\frac{64}{3}P_a{}^b\omega^3\omega_{;b}+\frac{8}{3}P_a{}^b\omega^2\mu_{;b}+\mu\omega_a{}^b\theta_{;b}-\frac{16}{3}\omega^2\omega_a{}^b\theta_{;b} \qquad \textbf{(1.69)}$$

> *TERM43r* := *rhs*(*step6l*) : *T*(%);

$$-\frac{5}{9}P_a{}^b\theta^2\mu_{;b}+\frac{2}{3}P_a{}^b\omega^2\mu_{;b}-\frac{1}{6}P_a{}^b\mu\mu_{;b}-\frac{1}{3}P_a{}^b\theta_{;b}\theta\mu-\frac{7}{6}\theta\omega_a{}^c\omega_c{}^b\theta_{;b}-\frac{1}{2}\omega_a{}^c\omega_c{}^b\theta_{;e}\omega^e{}_b+2\omega_a{}^c\omega_c{}^b\omega\omega_{;b}-\frac{1}{4}\omega_a{}^c\omega_c{}^b\mu_{;b}+\frac{1}{3}\omega_a{}^b\theta\mu_{;b} \qquad \textbf{(1.70)}$$

> *TERM43* := *TERM43l* − *TERM43r* = 0 : *T*(%);

$$-\frac{5}{3}P_a{}^b\theta_{;b}\theta\mu+\frac{112}{9}P_a{}^b\theta_{;b}\omega^2\theta+4P_a{}^b\mu\omega\omega_{;b}-\frac{1}{3}P_a{}^b\mu\mu_{;b}-\frac{64}{3}P_a{}^b\omega^3\omega_{;b}+2P_a{}^b\omega^2\mu_{;b}+\mu\omega_a{}^b\theta_{;b}-\frac{16}{3}\omega^2\omega_a{}^b\theta_{;b}+\frac{5}{9}P_a{}^b\theta^2\mu_{;b}+\frac{7}{6}\theta\omega_a{}^c\omega_c{}^b\theta_{;b}+\frac{1}{2}\omega_a{}^c\omega_c{}^b\theta_{;e}\omega^e{}_b-2\omega_a{}^c\omega_c{}^b\omega\omega_{;b}+\frac{1}{4}\omega_a{}^c\omega_c{}^b\mu_{;b}-\frac{1}{3}\omega_a{}^b\theta\mu_{;b}=0 \qquad \textbf{(1.71)}$$

> *TERM43a* := *collect*(*TERM43*, [theta[-*B*], *P*[-*a*, *b*], theta], `*distributed*`) : *T*(%);

$$\frac{1}{2}\omega_a{}^c\omega_c{}^b\theta_{;e}\omega^e{}_b-2\omega_a{}^c\omega_c{}^b\omega\omega_{;b}+\frac{1}{4}\omega_a{}^c\omega_c{}^b\mu_{;b}-\frac{1}{3}\omega_a{}^b\theta\mu_{;b} \qquad \textbf{(1.72)}$$

$$+\frac{5}{9}P_a{}^b\theta^2\mu_{;b}+\left(-\frac{5}{3}\mu+\frac{112}{9}\omega^2\right)\theta\theta_{;b}P_a{}^b+\frac{7}{6}\theta\omega_a{}^c\omega_c{}^b\theta_{;b}+\left(\mu\omega_a{}^b-\frac{16}{3}\omega^2\omega_a{}^b\right)\theta_{;b}+\left(4\mu\omega\omega_{;b}-\frac{1}{3}\mu\mu_{;b}-\frac{64}{3}\omega^3\omega_{;b}+2\omega^2\mu_{;b}\right)P_a{}^b=0$$

multiplying by ω^a

```
> TERM43b := expand(omega[a]·TERM43) : T(%);
```

$$-\frac{5}{3}\omega^aP_a{}^b\theta_{;b}\theta\mu+\frac{112}{9}\omega^aP_a{}^b\theta_{;b}\omega^2\theta+4\omega^aP_a{}^b\mu\omega\omega_{;b}-\frac{1}{3}\omega^aP_a{}^b\mu\mu_{;b}-\frac{64}{3}\omega^aP_a{}^b\omega^3\omega_{;b}+2\omega^aP_a{}^b\omega^2\mu_{;b}+\omega^a\mu\omega_a{}^b\theta_{;b}-\frac{16}{3}\omega^a\omega^2\omega_a{}^b\theta_{;b}+\frac{5}{9}\omega^aP_a{}^b\theta^2\mu_{;b}+\frac{7}{6}\omega^a\theta\omega_a{}^c\omega_c{}^b\theta_{;b}+\frac{1}{2}\omega^a\omega_a{}^c\omega_c{}^b\theta_{;e}\omega^e{}_b-2\omega^a\omega_a{}^c\omega_c{}^b\omega\omega_{;b}+\frac{1}{4}\omega^a\omega_a{}^c\omega_c{}^b\mu_{;b}-\frac{1}{3}\omega^a\omega_a{}^b\theta\mu_{;b}=0 \quad \textbf{(1.73)}$$

```
> TERM43c := expand(TEDS(omega[a]·omega[-a, c] = 0, TERM43b)) : T(%);
```

$$-\frac{5}{3}\omega^aP_a{}^b\theta_{;b}\theta\mu+\frac{112}{9}\omega^aP_a{}^b\theta_{;b}\omega^2\theta+4\omega^aP_a{}^b\mu\omega\omega_{;b}-\frac{1}{3}\omega^aP_a{}^b\mu\mu_{;b}-\frac{64}{3}\omega^aP_a{}^b\omega^3\omega_{;b}+2\omega^aP_a{}^b\omega^2\mu_{;b}+\omega^a\mu\omega_a{}^b\theta_{;b}-\frac{16}{3}\omega^a\omega^2\omega_a{}^b\theta_{;b}+\frac{5}{9}\omega^aP_a{}^b\theta^2\mu_{;b}-\frac{1}{3}\omega^a\omega_a{}^b\theta\mu_{;b}=0 \quad \textbf{(1.74)}$$

```
> TERM43d := expand(TEDS(omega[a]·omega[-a, b] = 0, TERM43c)) : T(%);
```

$$-\frac{5}{3}\omega^aP_a{}^b\theta_{;b}\theta\mu+\frac{112}{9}\omega^aP_a{}^b\theta_{;b}\omega^2\theta+4\omega^aP_a{}^b\mu\omega\omega_{;b}-\frac{1}{3}\omega^aP_a{}^b\mu\mu_{;b}-\frac{64}{3}\omega^aP_a{}^b\omega^3\omega_{;b}+2\omega^aP_a{}^b\omega^2\mu_{;b}+\frac{5}{9}\omega^aP_a{}^b\theta^2\mu_{;b}=0 \quad \textbf{(1.75)}$$

```
> TERM43e := expand(TEDS(omega[a]·P[-a, b] = omega[b], TERM43d)) : T(%);
```

$$-\frac{5}{3}\theta_{;b}\theta\mu\omega^b+\frac{112}{9}\theta_{;b}\omega^2\theta\omega^b+4\mu\omega\omega_{;b}\omega^b-\frac{1}{3}\mu\mu_{;b}\omega^b-\frac{64}{3}\omega^3\omega_{;b}\omega^b+2\omega^2\mu_{;b}\omega^b+\frac{5}{9}\theta^2\mu_{;b}\omega^b=0 \quad \textbf{(1.76)}$$

```
> TERM43f = collect(TERM43e, [theta, omega[-b], mu[-B]], `distributed`) : T(%);
```

$$TERM43f=\left(4\mu\omega\omega_{;b}\omega^b-\frac{64}{3}\omega^3\omega_{;b}\omega^b+\left(-\frac{5}{3}\theta_{;b}\mu\omega^b+\frac{112}{9}\theta_{;b}\omega^2\omega^b\right)\theta \quad \textbf{(1.77)}$$

$$+\left(-\frac{1}{3}\mu\,\omega^{\,b}+2\,\omega^2\,\omega^{\,b}\right)\mu_{;b}+\frac{5}{9}\theta^2\mu_{;b}\,\omega^{\,b}=0\Big)$$

Now from step 4:

> $eq1 : T(\%);$

$$\mu_{;b}\,\omega^{\,b}=8\,\omega\,\omega^{\,b}\,\omega_{;b} \qquad \textbf{(1.78)}$$

> $eq2 : T(\%);$

$$\theta_{;b}\,\mu\,\omega^{\,b}-\frac{16}{3}\,\theta_{;b}\,\omega^2\,\omega^{\,b}=\frac{1}{3}\,\theta\,\mu_{;b}\,\omega^{\,b} \qquad \textbf{(1.79)}$$

> $eq3 := \frac{1}{2}\cdot(rhs(eq1)=lhs(eq1)) : T(\%);$

$$4\,\omega\,\omega^{\,b}\,\omega_{;b}=\frac{1}{2}\,\mu_{;b}\,\omega^{\,b} \qquad \textbf{(1.80)}$$

> $TERM43g := expand(TEDS(\mathrm{eq1}, TERM43e)) : T(\%);$

$$\frac{4}{3}\,\mu\,\omega\,\omega_{;b}\,\omega^{\,b}-\frac{16}{3}\,\omega^3\,\omega_{;b}\,\omega^{\,b}+\frac{40}{9}\,\theta^2\,\omega\,\omega^{\,b}\,\omega_{;b}-\frac{5}{3}\,\theta_{;b}\,\theta\,\mu\,\omega^{\,b}+\frac{112}{9}\,\theta_{;b}\,\omega^2\,\theta\,\omega^{\,b}=0 \qquad \textbf{(1.81)}$$

****

REMAINING TERMS:

> $eq[43] := \left(\frac{1}{2}\right)\cdot\left(\mathrm{mu}+p-\left(\frac{16}{3}\right)\cdot\omega^2\right)\cdot\mathrm{omega}[-a,b]\cdot\mathrm{theta}[-B]+\mathrm{theta}\cdot\left(\left(\frac{112}{9}\right)\cdot\mathrm{omega}\cdot\mathrm{omega}-\left(\frac{5}{3}\right)\cdot(\mathrm{mu}+p)\right)\cdot P[-a,b]\cdot\mathrm{theta}[-B]+\left(\left(\frac{5}{9}\right)\cdot\mathrm{theta}\cdot\mathrm{theta}-\left(\frac{2}{3}\right)\cdot\mathrm{omega}\cdot\mathrm{omega}+\left(\frac{1}{6}\right)\cdot(\mathrm{mu}+3\cdot p)\right)P[-a,b]\cdot\mathrm{mu}[-B]+\left(\frac{7}{6}\right)\cdot\mathrm{theta}\cdot\mathrm{omega}[-a,c]\cdot\mathrm{omega}[-c,b]\cdot\mathrm{theta}[-B]-\left(\frac{1}{4}\right)\cdot\mathrm{omega}[-a,d]\cdot\mathrm{omega}[-d,c]\cdot\mathrm{omega}[-c,b]\cdot\mathrm{theta}[-B]=0 : T(\%);$

$$\frac{1}{2}\left(\mu+p-\frac{16}{3}\,\omega^2\right)\omega_a^{\ b}\,\theta_{;b}+\theta\left(\frac{112}{9}\,\omega^2-\frac{5}{3}\,\mu-\frac{5}{3}\,p\right)P_a^{\ b}\,\theta_{;b}+\left(\frac{5}{9}\,\theta^2-\frac{2}{3}\,\omega^2+\frac{1}{6}\,\mu+\frac{1}{2}\,p\right)P_a^{\ b}\,\mu_{;b}+\frac{7}{6}\,\theta\,\omega_a^{\ c}\,\omega_c^{\ b}\,\theta_{;b}-\frac{1}{4}\,\omega_a^{\ d}\,\omega_d^{\ c}\,\omega_c^{\ b}\,\theta_{;b}=0 \qquad \textbf{(1.82)}$$

> $T(subs(p=0, expand(eq[43])));$

$$\frac{1}{2}\,\mu\,\omega_a^{\ b}\,\theta_{;b}-\frac{8}{3}\,\omega^2\,\omega_a^{\ b}\,\theta_{;b}+\frac{112}{9}\,P_a^{\ b}\,\theta_{;b}\,\omega^2\,\theta-\frac{5}{3}\,P_a^{\ b}\,\theta_{;b}\,\theta\,\mu+\frac{5}{9}\,P_a^{\ b}\,\theta^2\,\mu_{;b}-\frac{2}{3}\,P_a^{\ b}\,\omega^2\,\mu_{;b}+\frac{1}{6}\,P_a^{\ b}\,\mu\,\mu_{;b}+\frac{7}{6}\,\theta\,\omega_a^{\ c}\,\omega_c^{\ b}\,\theta_{;b}-\frac{1}{4}\,\omega_a^{\ d}\,\omega_d^{\ c}\,\omega_c^{\ b}\,\theta_{;b}=0 \qquad \textbf{(1.83)}$$

> T(TERM43);

$$-\frac{5}{3} P_a{}^b \theta_{;b} \theta \mu + \frac{112}{9} P_a{}^b \theta_{;b} \omega^2 \theta + 4 P_a{}^b \mu \omega \omega_{;b} - \frac{1}{3} P_a{}^b \mu \mu_{;b} - \frac{64}{3} P_a{}^b \omega^3 \omega_{;b} + 2 P_a{}^b \omega^2 \mu_{;b} + \mu \omega_a{}^b \theta_{;b} - \frac{16}{3} \omega^2 \omega_a{}^b \theta_{;b} + \frac{5}{9} P_a{}^b \theta^2 \mu_{;b} + \frac{7}{6} \theta \omega_a{}^c \omega_c{}^b \theta_{;b} + \frac{1}{2} \omega_a{}^c \omega_c{}^b \theta_{;e} \omega^e{}_b - 2 \omega_a{}^c \omega_c{}^b \omega \omega_{;b} + \frac{1}{4} \omega_a{}^c \omega_c{}^b \mu_{;b} - \frac{1}{3} \omega_a{}^b \theta \mu_{;b} = 0 \quad \textbf{(1.84)}$$

> REMAINDER := expand(lhs(subs(p = 0, eq[43]))) − lhs(TERM43) : T(%);

$$-\frac{1}{2} \mu \omega_a{}^b \theta_{;b} + \frac{8}{3} \omega^2 \omega_a{}^b \theta_{;b} - \frac{8}{3} P_a{}^b \omega^2 \mu_{;b} + \frac{1}{2} P_a{}^b \mu \mu_{;b} - \frac{1}{4} \omega_a{}^d \omega_d{}^c \omega_c{}^b \theta_{;b} - 4 P_a{}^b \mu \omega \omega_{;b} + \frac{64}{3} P_a{}^b \omega^3 \omega_{;b} - \frac{1}{2} \omega_a{}^c \omega_c{}^b \theta_{;e} \omega^e{}_b + 2 \omega_a{}^c \omega_c{}^b \omega \omega_{;b} - \frac{1}{4} \omega_a{}^c \omega_c{}^b \mu_{;b} + \frac{1}{3} \omega_a{}^b \theta \mu_{;b} \quad \textbf{(1.85)}$$

to prove the equation, we can try to show that REMAINDER=0

> eq[40] : T(%);

$$-8 \omega P_a{}^b \omega_{;b} + P_a{}^b \mu_{;b} + \omega_a{}^b \theta_{;b} = 0 \quad \textbf{(1.86)}$$

> temp := isolate(eq[40], P[−a, b]·mu[−B]) : T(%);

$$P_a{}^b \mu_{;b} = 8 \omega P_a{}^b \omega_{;b} - \omega_a{}^b \theta_{;b} \quad \textbf{(1.87)}$$

> rem1 := expand(TEDS(temp, REMAINDER = 0)) : T(%);

$$-\mu \omega_a{}^b \theta_{;b} + \frac{16}{3} \omega^2 \omega_a{}^b \theta_{;b} - \frac{1}{4} \omega_a{}^d \omega_d{}^c \omega_c{}^b \theta_{;b} - \frac{1}{2} \omega_a{}^c \omega_c{}^b \theta_{;e} \omega^e{}_b + 2 \omega_a{}^c \omega_c{}^b \omega \omega_{;b} - \frac{1}{4} \omega_a{}^c \omega_c{}^b \mu_{;b} + \frac{1}{3} \omega_a{}^b \theta \mu_{;b} = 0 \quad \textbf{(1.88)}$$

> eq[41] : T(%);

$$-8 \omega \omega_{;b} \omega_a{}^b + \omega_a{}^c \omega_c{}^b \theta_{;b} + \omega_a{}^b \mu_{;b} = 0 \quad \textbf{(1.89)}$$

> step42b : T(%);

$$\left(\mu - \frac{16}{3} \omega^2\right) \theta_{;b} \omega_a{}^b = \frac{1}{2} \omega_a{}^d \omega_d{}^c \omega_c{}^b \theta_{;b} + \frac{1}{3} \omega_a{}^b \theta \mu_{;b} \quad \textbf{(1.90)}$$

> temp := mu·omega[−a, b]·theta[−B] = $\frac{1}{2}$ ·omega[−a, d]·omega[−d, c]·omega[−c, b]·theta[−B] + $\frac{1}{3}$ ·omega[−a, b]·theta·mu[−B] + $\frac{16}{3}$ ·omega·omega·omega[−a, b]·theta[−B] : T(%);

$$\mu \omega_a{}^b \theta_{;b} = \frac{1}{2} \omega_a{}^d \omega_d{}^c \omega_c{}^b \theta_{;b} + \frac{1}{3} \omega_a{}^b \theta \mu_{;b} + \frac{16}{3} \omega^2 \omega_a{}^b \theta_{;b} \quad \textbf{(1.91)}$$

```
> rem2 := expand(TEDS(temp, rem1)) : T(%);
```

$$-\frac{3}{4}\,\omega_a{}^d\omega_d{}^c\omega_c{}^b\theta_{;b}-\frac{1}{2}\,\omega_a{}^c\omega_c{}^b\theta_{;e}\omega^e{}_b+2\,\omega_a{}^c\omega_c{}^b\omega\,\omega_{;b}-\frac{1}{4}\,\omega_a{}^c\omega_c{}^b\mu_{;b}=0 \qquad \textbf{(1.92)}$$

```
> temp := omega[-a, c]·omega[-c, b]·omega[-b, e]·theta[-E] = omega[-a, d]·omega[-d, c]
    ·omega[-c, b]·theta[-B] : T(%);
```

$$\omega_a{}^c\omega_c{}^b\omega_b{}^e\theta_{;e}=\omega_a{}^d\omega_d{}^c\omega_c{}^b\theta_{;b} \qquad \textbf{(1.93)}$$

```
> rem3 := expand(TEDS(temp, rem2)) : T(%);
```

$$-\frac{3}{4}\,\omega_a{}^d\omega_d{}^c\omega_c{}^b\theta_{;b}-\frac{1}{2}\,\omega_a{}^c\omega_c{}^b\theta_{;e}\omega^e{}_b+2\,\omega_a{}^c\omega_c{}^b\omega\,\omega_{;b}-\frac{1}{4}\,\omega_a{}^c\omega_c{}^b\mu_{;b}=0 \qquad \textbf{(1.94)}$$

from previous file

```
> read "Seneqs3d" :
> eq[41 c] : T(%);
```

$$-8\,\omega_a{}^c\omega_c{}^b\omega\,\omega_{;b}+\omega_a{}^c\omega_c{}^d\omega_d{}^b\theta_{;b}+\omega_a{}^c\omega_c{}^b\mu_{;b}=0 \qquad \textbf{(1.95)}$$

so we have:

```
> rem4 := rem3 + 1/4·eq[41 c] : T(%);
```

$$-\frac{3}{4}\,\omega_a{}^d\omega_d{}^c\omega_c{}^b\theta_{;b}-\frac{1}{2}\,\omega_a{}^c\omega_c{}^b\theta_{;e}\omega^e{}_b+\frac{1}{4}\,\omega_a{}^c\omega_c{}^d\omega_d{}^b\theta_{;b}=0 \qquad \textbf{(1.96)}$$

```
> temp := omega[-a, d]·omega[-d, c]·omega[-c, b]·theta[-B] = omega[-a, c]·omega[-c, d]
    ·omega[-d, b]·theta[-B] : T(%);
```

$$\omega_a{}^d\omega_d{}^c\omega_c{}^b\theta_{;b}=\omega_a{}^c\omega_c{}^d\omega_d{}^b\theta_{;b} \qquad \textbf{(1.97)}$$

```
> rem5 := expand(TEDS(temp, rem4)) : T(%);
```

$$-\frac{1}{2}\,\omega_a{}^c\omega_c{}^d\omega_d{}^b\theta_{;b}-\frac{1}{2}\,\omega_a{}^c\omega_c{}^b\theta_{;e}\omega^e{}_b=0 \qquad \textbf{(1.98)}$$

```
> temp := omega[-a, c]·omega[-c, b]·omega[e, -b]·theta[-E] = -omega[-a, c]·omega[-c, d]
    ·omega[-d, b]·theta[-B] : T(%);
```

$$\omega_a{}^c\omega_c{}^b\theta_{;e}\omega^e{}_b=-\omega_a{}^c\omega_c{}^d\omega_d{}^b\theta_{;b} \qquad \textbf{(1.99)}$$

```
> rem6 := expand(TEDS(temp, rem5)) : T(%);
```

"LHS is a constant, RHS is a tensor - suggest reverse the equation"

$$0=0 \qquad \textbf{(1.100)}$$

thus completing the proof

****
****
****

****

and so we have shown:

```
> eq[43] := (1/2)·(mu + p - (16/3)·ω^2)·omega[-a, b]·theta[-B] + theta·(((112/9)
    ·omega·omega - (5/3)·(mu + p))·P[-a, b]·theta[-B] + (((5/9)·theta·theta - (2/3)
    ·omega·omega + (1/6)·(mu + 3·p))P[-a, b]·mu[-B] + (7/6)·theta·omega[-a, c]
    ·omega[-c, b]·theta[-B] - (1/4)·omega[-a, d]·omega[-d, c]·omega[-c, b]·theta[
    -B] = 0 : T(%);
```

$$\frac{1}{2}\left(\mu+p-\frac{16}{3}\omega^2\right)\omega_a{}^b\theta_{;b}+\theta\left(\frac{112}{9}\omega^2-\frac{5}{3}\mu-\frac{5}{3}p\right)P_a{}^b\theta_{;b}+\left(\frac{5}{9}\theta^2-\frac{2}{3}\omega^2+\frac{1}{6}\mu+\frac{1}{2}p\right)P_a{}^b\mu_{;b}+\frac{7}{6}\theta\,\omega_a{}^c\omega_c{}^b\theta_{;b}-\frac{1}{4}\omega_a{}^d\omega_d{}^c\omega_c{}^b\theta_{;b}=0 \quad \mathbf{(1.101)}$$

```
>

>

> PrintSubArray(eq, 1, 43, yes);
```

$$1,\ T_{ab}=\rho\, u_a u_b$$

$$2,\ P_{ab}=u\, u_{ab}+g_{ab}$$

$$3,\ P^a{}_b u^b=0$$

$$4,\ dX^a=u^b X^a{}_{;b}$$

$$5,\ du^a=u^b u^a{}_{;b}$$

$$6,\ u_{a;b}=\frac{1}{3}\theta P_{ab}+\sigma_{ab}+\omega_{ab}-du_a u_b$$

$$7,\ \theta=u^a{}_{;a}$$

$$8,\ \sigma_{ab}=\frac{1}{2}P_a{}^c P_b{}^d u_{c;d}+\frac{1}{2}P_b{}^c P_a{}^d u_{c;d}-\frac{1}{3}\theta P_{ab}$$

$$9,\ \omega_{ab}=\frac{1}{2}P_a{}^c P_b{}^d u_{c;d}-\frac{1}{2}P_b{}^c P_a{}^d u_{c;d}$$

$$10,\ \omega^a=\frac{1}{2}\eta^{abcd}u_b\omega_{cd}$$

$$11,\ \omega_{ab}=\eta_{abef}\omega^e u^f$$

$$12,\ \omega^2=\frac{1}{2}\omega^{ab}\omega_{ab}$$

13, "iff(iff(omega[-a,-b] = 0,omega[-a]),omega = 0)"

$$14, \omega_{a}{}^{c}\omega_{c}{}^{b} = -\omega^{2} P_{a}{}^{b} + \omega^{b}\omega_{a}$$

$$15, \frac{1}{2} u_{b;a} - \frac{1}{2} u_{a;b} = \frac{1}{2} du_{a} u_{b} - \frac{1}{2} du_{b} u_{a} + \omega^{a\ b}$$

$$16, -\frac{1}{6} u_{c} u_{a;b} + \frac{1}{6} u_{c} u_{b;a} + \frac{1}{6} u_{b} u_{a;c} - \frac{1}{6} u_{b} u_{c;a} - \frac{1}{6} u_{a} u_{b;c} + \frac{1}{6} u_{a} u_{c;b} = 0$$

$$17, \sigma_{a\,b} = 0$$

$$18, u_{a;b} = \frac{1}{3} \theta P_{a\,b} + \omega_{a\,b}$$

$$19, u^{a}{}_{;c;d} - u^{a}{}_{;d;c} = R^{a}{}_{b\,c\,d} u^{b}$$

$$20, dottheta + \frac{1}{3} \theta^{2} - 2\,\omega^{2} + \frac{1}{2} \mu = 0$$

$$21, P_{a}{}^{c} P_{b}{}^{d} \omega_{c\,d;f} u^{f} + \frac{2}{3} \theta\,\omega_{a\,b} = 0$$

$$22, \omega_{a}\omega_{b} - \frac{1}{3} P_{a\,b}\omega^{2} + E_{a\,b} = 0$$

$$23, E_{a\,b} = C_{a\,b\,c\,d} u^{c} u^{d}$$

$$24, H_{a\,b} = \frac{1}{2} \eta_{a\,e}{}^{c\ d} C_{c\,d\,b\,f} u^{e} u^{f}$$

$$25, P^{a}{}_{b}\,\omega^{b}{}_{;f} u^{f} + \frac{2}{3} \theta\,\omega^{a} = 0$$

$$26, 2\,P^{a\ b}\theta_{;b} + 3\,P^{a}{}_{b}\,\omega^{b\ d}{}_{;d} = 0$$

$$27, \omega^{a}{}_{;a} = 2\,du^{a}\omega_{a}$$

$$28, H_{a\,b} = \frac{1}{2} P_{a}{}^{c} P_{b}{}^{d}\,\omega^{d\ ;c} + \frac{1}{2} P_{b}{}^{c} P_{a}{}^{d}\,\omega^{d\ ;c}$$

$$29, \omega_{a\,b}\,\omega^{b\ c}{}_{;c} = P_{a}{}^{b}\,\omega^{c}\,\omega_{b;c} - P_{a}{}^{b}\,\omega^{c}\,\omega_{c;b}$$

$$30, \mu\,\theta + dotmu = 0$$

$$31, (\mu + p)\, du^{a} + P^{a\ b} p_{;b}$$

$$32, du^{a} = 0$$

$$33, u_{a} = -\frac{f_{;a}}{fdot}$$

$$34, \mu = (c1 - 1)\, p + c2\,\omega^{2}$$

$$35, dotomega_{a\,b} = -\frac{2}{3} \theta\,\omega_{a\,b}$$

$$36,\ dotomega = -\frac{2}{3}\,\theta\,\omega$$

$$37,\ \theta\left(c1p - \frac{1}{3}\,c2\,\omega^2\right) = 0$$

$$38,\ \frac{\partial}{\partial t}\left(P^{a\ b} f_{;b}\right) = P^{a\ b} f``_{;b} + \omega^{a\ b} f_{;b} - \frac{1}{3}\,\theta\,P^{a\ b} f_{;b}$$

$$39,\ -3\,P_a{}^b\,\omega^c\,\omega_{b\,;c} - 13\,P_a{}^b\,\omega^c\,\omega_{c\,;b} + 2\,P_a{}^b\,\mu_{;b} = 0$$

$$40,\ -8\,\omega\,P_a{}^b\,\omega_{;b} + P_a{}^b\,\mu_{;b} + \omega_a{}^b\,\theta_{;b} = 0$$

$$41,\ -8\,\omega\,\omega_{;b}\,\omega_a{}^b + \omega_a{}^c\,\omega_c{}^b\,\theta_{;b} + \omega_a{}^b\,\mu_{;b} = 0$$

$$42,\ \omega_a{}^b\,dottheta_{;b} + 4\,\omega\,\omega_{;b}\,\omega_a{}^b + \frac{1}{2}\,\omega_a{}^b\,\mu_{;b} + 8\,\omega\,\theta\,P^2_{-a,\,b}\,\omega_{;b} - P_a{}^b\,\theta_{;b}\,\mu + \frac{16}{3}\,\omega^2\,P^2_{-a,\,b}\,\theta_{;b} + \left(\omega_a{}^b\,\theta - \frac{1}{2}\,\omega_a{}^c\,\omega_c{}^b\right)\theta_{;b} + \left(\frac{1}{3}\,\theta\,\mu_{;b} - 8\,\omega\,\omega_{;b}\,\omega_a{}^b - \frac{40}{3}\,\omega\,\theta\,\omega_{;b}\right)P_a{}^b = 0$$

$$43,\ \frac{1}{2}\left(\mu + p - \frac{16}{3}\,\omega^2\right)\omega_a{}^b\,\theta_{;b} + \theta\left(\frac{112}{9}\,\omega^2 - \frac{5}{3}\,\mu - \frac{5}{3}\,p\right)P_a{}^b\,\theta_{;b} + \left(\frac{5}{9}\,\theta^2 - \frac{2}{3}\,\omega^2 + \frac{1}{6}\,\mu + \frac{1}{2}\,p\right)P_a{}^b\,\mu_{;b} + \frac{7}{6}\,\theta\,\omega_a{}^c\,\omega_c{}^b\,\theta_{;b} - \frac{1}{4}\,\omega_a{}^d\,\omega_d{}^c\,\omega_c{}^b\,\theta_{;b} = 0 \quad \textbf{(1.102)}$$

```
> save eq, "Seneqs4" :
>
```