```
> restart;
> with(Riemann):with(Canon):
> with(TensorPack) : CDF(0) : CDS(index) :
```

Chapter XX
Tensor analysis using indices - Senovilla et al. - Shearfree for dust
page 2
if $\sigma_{a\,b}$=0 =>ωΘ=0
Author: Peter Huf
file 2
eq38

```
> read "EFE" : read "SFE" :read "fids" :read "Seneqs2e" :
```

****

Equation 38

****

```
> eq[38] := Diff(P[a, b]·f[-B], t) = P[a, b]·f`'`[-B] + omega[a, b]·f[-B] - (1/3)· theta
      ·P[a, b]·f[-B] : T(%);
```

$$\frac{\partial}{\partial t}\left(P^{a\ b} f_{;b}\right) = P^{a\ b} f`'`_{;b} + \omega^{a\ b} f_{;b} - \frac{1}{3}\,\theta\,P^{a\ b} f_{;b} \qquad \textbf{(1.1)}$$

We make use of:

```
> eq[18] := u[-a,-B] = (1/3)·P[-a,-b]·theta + omega[-a,-b] : T(%);
```

$$u_{a\,;b} = \frac{1}{3}\,\theta\,P_{a\,b} + \omega_{a\,b} \qquad \textbf{(1.2)}$$

which is derived directly from SSSeq6 where shear = 0 and acceleration=0:

```
>
```

proof of eq38 :

The lhs of eq38 can be written as:

```
> eq[38 l] := dotT(P[a, b]·f[-B]) : T(%);
```

$$P^{a\ b} dotf_{;b} + dotP^{a\ b} f_{;b} \qquad \textbf{(1.3)}$$

where dotP[a,b]=0 dince du=0

```
> eq[38 l] := TEDS(dotP[a, b] = 0, eq[38 l]) : T(%);
```

$$P^{a\ b} dotf_{;b} \qquad \textbf{(1.4)}$$

Now subs for P

```
> temp := P[a, b] = g[a, b] + u[a, ]·u[b] : T(%);
```

$$P^{a\ b} = u^{a} u^{b} + g^{a\ b} \qquad \textbf{(1.5)}$$

```
> eq[38 l] := Absorbg(expand(TEDS(temp, eq[38 l]))) : T(%);
```

$$dotf_{;b}\, u^{a}\, u^{b} + dotf^{;a} \quad \textbf{(1.6)}$$

>

>

Now let us look at the rhs of eq38:

```
> eq[38] := Diff(P[a, b]·f[-B], t) = P[a, b]·f`'`[-B] + omega[a, b]·f[-B] - (1/3)· theta
    ·P[a, b]·f[-B] : T(%);
```

$$\frac{\partial}{\partial t}\left(P^{a\,b} f_{;b}\right) = P^{a\,b} f`'`_{;b} + \omega^{a\,b} f_{;b} - \frac{1}{3}\,\theta\, P^{a\,b} f_{;b} \quad \textbf{(1.7)}$$

We can also write this as:

```
> eq[38] := Diff(P[a, b]·f[-B], t) = P[a, b]·fdot[-B] + omega[a, b]·f[-B] - (1/3)· theta
    ·P[a, b]·f[-B] : T(%);
```

$$\frac{\partial}{\partial t}\left(P^{a\,b} f_{;b}\right) = P^{a\,b} fdot_{;b} + \omega^{a\,b} f_{;b} - \frac{1}{3}\,\theta\, P^{a\,b} f_{;b} \quad \textbf{(1.8)}$$

where fdot=f`'` is the time derivative of f

>

So the rhs of eq38 is:

```
> eq[38 r] := factor(rhs(eq[38])) : T(%);
```

$$P^{a\,b} fdot_{;b} + \omega^{a\,b} f_{;b} - \frac{1}{3}\,\theta\, P^{a\,b} f_{;b} \quad \textbf{(1.9)}$$

Now the differential term for fdot[-B] can be expressed as :

```
> temp := fdot[-B] = f[-E,-B]·u[e] + f[-E]·u[e,-B] : T(%);
```

$$fdot_{;b} = f_{;e}\, u^{e}{}_{;b} + f_{;e\,;b}\, u^{e} \quad \textbf{(1.10)}$$

and substituting this into the rhs of eq38 gives:

```
> temp2 := expand(TEDS(temp, eq[38 r])) : T(%);
```

$$P^{a\,b} f_{;e}\, u^{e}{}_{;b} + P^{a\,b} f_{;e\,;b}\, u^{e} + \omega^{a\,b} f_{;b} - \frac{1}{3}\,\theta\, P^{a\,b} f_{;b} \quad \textbf{(1.11)}$$

we substitute the identity eq18 above (with suitable indices):

```
> temp3 := expand(subs(a =-e, eq[18])) : T(%);
```

$$u^{e}{}_{;b} = \frac{1}{3}\,\theta\, P^{e}{}_{b} + \omega^{e}{}_{b} \quad \textbf{(1.12)}$$

```
> temp4 := expand(temp3·f[-E]) : T(%);
```

$$f_{;e}\, u^{e}{}_{;b} = \frac{1}{3}\, f_{;e}\, \theta\, P^{e}{}_{b} + f_{;e}\, \omega^{e}{}_{b} \quad \textbf{(1.13)}$$

```
> temp5 := expand(TEDS(temp4, temp2)) : T(%);
```

$$P^{a\,b} f_{;e\,;b}\, u^{e} + \omega^{a\,b} f_{;b} - \frac{1}{3}\,\theta\, P^{a\,b} f_{;b} + \frac{1}{3}\,\theta\, P^{a\,b} P^{e}{}_{b} f_{;e} + P^{a\,b} f_{;e}\, \omega^{e}{}_{b} \quad \textbf{(1.14)}$$

>

NOW THE FOLLOWING IS ASSUMED TO BE TRUE FOR A SCALAR FUNCTION (f) IF THE SPACETIME IS TORSION-FREE (REFERENCE BERTSCHINGER, 2002):

> *temp6* := *f*[-*E*,-*B*]=*f*[-*B*,-*E*] : *T*(%);

$$f_{;e\,;b} = f_{;b\,;e} \quad \textbf{(1.15)}$$

And so we substitute this into the rhs term of eq38:

> *temp7* := *TEDS*(*temp6*, *temp5*) : *T*(%);

$$P^{a\ b} u^{e} f_{;b\,;e} + \omega^{a\ b} f_{;b} - \frac{1}{3}\,\theta\,P^{a\ b} f_{;b} + \frac{1}{3}\,\theta\,P^{a\ b} P^{e}{}_{b} f_{;e} + P^{a\ b} f_{;e}\,\omega^{e}{}_{b} \quad \textbf{(1.16)}$$

Now we substitute basic identities:

> *temp8* := *Absorbg*(*TELS*(*P*[*a*, *b*] = *g*[*a*, *b*] + *u*[*a*]·*u*[*b*], *TEDS*(*P*[*a*, *b*]·*P*[*e*,-*b*] = *P*[*a*, *e*], *temp7*))) : *T*(%);

$$-\frac{1}{3}\,\theta f_{;b} u^{a} u^{b} + f_{;e}\,\omega^{e}{}_{b} u^{a} u^{b} + f_{;b\,;e} u^{a} u^{b} u^{e} + f_{;e}\,\omega^{e\ a} + u^{e} f^{;a}{}_{;e} + \omega^{a\ b} f_{;b} + \frac{1}{3}\,\theta f_{;e} u^{a} u^{e} \quad \textbf{(1.17)}$$

> *temp9* := *expand*(*TEDS*(*f*[-*E*]·*u*[*e*]=*fdot*, *expand*(*TEDS*(*f*[-*B*]·*u*[*b*]=*fdot*, *temp8*)))) : *T*(%);

$$f_{;e}\,\omega^{e}{}_{b} u^{a} u^{b} + f_{;b\,;e} u^{a} u^{b} u^{e} + \omega^{a\ b} f_{;b} + f_{;e}\,\omega^{e\ a} + u^{e} f^{;a}{}_{;e} \quad \textbf{(1.18)}$$

> *temp10* := *TEDS*(omega[*e*,-*b*]·*u*[*b*] = 0, *temp9*) : *T*(%);

$$f_{;b\,;e} u^{a} u^{b} u^{e} + \omega^{a\ b} f_{;b} + f_{;e}\,\omega^{e\ a} + u^{e} f^{;a}{}_{;e} \quad \textbf{(1.19)}$$

> *temp11* := *TEDS*(*f*[*A*,-*E*]·*u*[*e*]=*dotf*[*A*], *TEDS*(*f*[-*B*,-*E*]·*u*[*e*]=*dotf*[-*B*], *temp10*)) : *T*(%);

$$dotf_{;b} u^{a} u^{b} + \omega^{a\ b} f_{;b} + f_{;e}\,\omega^{e\ a} + dotf^{;a} \quad \textbf{(1.20)}$$

> *temp12* := *TEDS*(omega[*a*, *b*]·*f*[-*B*]=-omega[*e*, *a*]·*f*[-*E*], *temp11*) : *T*(%);

$$dotf_{;b} u^{a} u^{b} + dotf^{;a} \quad \textbf{(1.21)}$$

and arrive at the same term as for the lhs of eq38, shown in *eq*[38 *l*] :

> *eq*[38 *l*] : *T*(%);

$$dotf_{;b} u^{a} u^{b} + dotf^{;a} \quad \textbf{(1.22)}$$

thus completing the proof of eq38.

>

>

> *eq*[38] := *Diff*(*P*[*a*, *b*]·*f*[-*B*], *t*) = *P*[*a*, *b*]·*fdot*[-*B*] + omega[*a*, *b*]·*f*[-*B*] − $\left(\frac{1}{3}\right)$· theta ·*P*[*a*, *b*]·*f*[-*B*] : *T*(%);

$$\frac{\partial}{\partial t}\left(P^{a\ b} f_{;b}\right) = P^{a\ b} fdot_{;b} + \omega^{a\ b} f_{;b} - \frac{1}{3}\,\theta\,P^{a\ b} f_{;b} \quad \textbf{(1.23)}$$

> **save** *eq*, "Seneqs2f" :

go to page 3

```
> read "Seneqs2f";
```

$$eq := table\Big(\Big[1 = (TensorPack{:}\text{-}T_{-a,\,-b} = \rho\, u_{-a} u_{-b}),\ 2 = (P_{-a,\,-b} = u_{-a} u_{-b} + g_{-a,\,-b}),\ 3 = (P_{a,\,-b} u_{b} = 0),\ 4 = (dX_{a} = u_{b} X_{a,\,-B}),\ 5 = (du_{a} = u_{b} u_{a,\,-B}),\ 6 = \Big(u_{-a,\,-B} = \frac{1}{3}\,\theta\, P_{-a,\,-b} + \sigma_{-a,\,-b} + \omega_{-a,\,-b} - du_{-a} u_{-b}\Big),\ 35\,a = (P_{-a,\,c} P_{-b,\,d}\, dotomega_{-c,\,-d} = dotomega_{-a,\,-b}),\ 7 = (\theta = u_{a,\,-A}),\ 9 = \Big(\omega_{-a,\,-b} = \frac{1}{2}\, P_{-a,\,c} P_{-b,\,d} u_{-c,\,-D} - \frac{1}{2}\, P_{-b,\,c} P_{-a,\,d} u_{-c,\,-D}\Big),\ 8 = \Big(\sigma_{-a,\,-b} = \frac{1}{2}\, P_{-a,\,c} P_{-b,\,d} u_{-c,\,-D} + \frac{1}{2}\, P_{-b,\,c} P_{-a,\,d} u_{-c,\,-D} - \frac{1}{3}\,\theta\, P_{-a,\,-b}\Big),\ 11 = (\omega_{-a,\,-b} = \eta_{-a,\,-b,\,-e,\,-f}\,\omega_{e} u_{f}),\ 10 = \Big(\omega_{a} = \frac{1}{2}\,\eta_{a,\,b,\,c,\,d}\, u_{-b}\,\omega_{-c,\,-d}\Big),\ 13 = \text{"iff(iff(omega[-a,-b] = 0,omega[-a]),omega = 0)"},\ 12 = \Big(\omega^{2} = \frac{1}{2}\,\omega_{a,\,b}\,\omega_{-a,\,-b}\Big),\ 15 = \Big(\frac{1}{2}\, u_{-b,\,-A} - \frac{1}{2}\, u_{-a,\,-B} = \frac{1}{2}\, du_{-a} u_{-b} - \frac{1}{2}\, du_{-b} u_{-a} + \omega_{a,\,b}\Big),\ 14 = (\omega_{-a,\,c}\,\omega_{-c,\,b} = -\omega^{2} P_{-a,\,b} + \omega_{b}\,\omega_{-a}),\ 18 = \Big(u_{-a,\,-B} = \frac{1}{3}\,\theta\, P_{-a,\,-b} + \omega_{-a,\,-b}\Big),\ 19 = (u_{a,\,-C,\,-D} - u_{a,\,-D,\,-C} = R_{a,\,-b,\,-c,\,-d} u_{b}),\ 16 = \Big(-\frac{1}{6}\, u_{-c} u_{-a,\,-B} + \frac{1}{6}\, u_{-c} u_{-b,\,-A} + \frac{1}{6}\, u_{-b} u_{-a,\,-C} - \frac{1}{6}\, u_{-b} u_{-c,\,-A} - \frac{1}{6}\, u_{-a} u_{-b,\,-C} + \frac{1}{6}\, u_{-a} u_{-c,\,-B} = 0\Big),\ 17 = (\sigma_{-a,\,-b} = 0),\ 22 = \Big(\omega_{-a}\,\omega_{-b} - \frac{1}{3}\, P_{-a,\,-b}\,\omega^{2} + E_{-a,\,-b} = 0\Big),\ 23 = (E_{-a,\,-b} = C_{-a,\,-b,\,-c,\,-d} u_{c} u_{d}),\ 20 = \Big(dottheta + \frac{1}{3}\,\theta^{2} - 2\,\omega^{2} + \frac{1}{2}\,\mu = 0\Big),\ 35\,b = \Big(dotomega_{-a,\,-b} = -\frac{2}{3}\,\theta\,\omega_{-a,\,-b}\Big),\ 21 = \Big(P_{-a,\,c} P_{-b,\,d}\,\omega_{-c,\,-d,\,-F} u_{f} + \frac{2}{3}\,\theta\,\omega_{-a,\,-b} = 0\Big),\ 27 = (\omega_{a,\,-A} = 2\, du_{a}\,\omega_{-a}),\ 26 = 2\, P_{a,\,b}\,\theta_{-B} + 3\, P_{a,\,-b}\,\omega_{b,\,d,\,-D},\ 36\,b = \Big(dotomega_{a} = -\frac{2}{3}\,\theta\,\omega_{a}\Big),\ 25 = \Big(P_{a,\,-b}\,\omega_{b,\,-F} u_{f} + \frac{2}{3}\,\theta\,\omega_{a} = 0\Big),\ 24 = \Big(H_{-a,\,-b} = \frac{1}{2}\,\eta_{-a,\,-e,\,c,\,d}\, C_{-c,\,-d,\,-b,\,-f} u_{e} u_{f}\Big),\ 31 = (\mu + p)\, du_{a} + P_{a,\,b}\, p_{-B},\ 30 = (\mu\,\theta + dotmu \tag{1.24}$$

$= 0), 29 = \left(\omega_{-a,\ -n} \omega_{n,\ m,\ -M} = \omega^2 du_{-a} + P_{-a,\ b} \omega_c \omega_{-b,\ -C} - P_{-a,\ b} \omega_c \omega_{-c,\ -B}$

$- du_{-p} \omega_p \omega_{-a} \right), 28 = \left(H_{-a,\ -b} = \frac{1}{2} P_{-a,\ c} P_{-b,\ d} \omega_{d,\ C} + \frac{1}{2} P_{-b,\ c} P_{-a,\ d} \omega_{d,\ C} \right), 36$

$= \left(dotomega = -\frac{2}{3} \theta \omega \right), 10\ b = \left(\eta_{-f,\ -g,\ -a,\ -e} \omega_a u_e \right.$

$\left. = \frac{1}{2} \eta_{-f,\ -g,\ -a,\ -e} \eta_{a,\ b,\ c,\ d} u_{-b} \omega_{-c,\ -d} u_e \right), 37 = \left(\theta \left(c1p - \frac{1}{3} c2\, \omega^2 \right) = 0 \right), 38$

$= \left(\frac{\partial}{\partial t} \left(P_{a,\ b} f_{-B} \right) = P_{a,\ b} fdot_{-B} + \omega_{a,\ b} f_{-B} - \frac{1}{3} \theta P_{a,\ b} f_{-B} \right), 16\ a = \left(-\frac{1}{6} u_{-c} u_{-a,\ -B} \right.$

$+ \frac{1}{6} u_{-c} u_{-b,\ -A} + \frac{1}{6} u_{-b} u_{-a,\ -C} - \frac{1}{6} u_{-b} u_{-c,\ -A} - \frac{1}{6} u_{-a} u_{-b,\ -C} + \frac{1}{6} u_{-a} u_{-c,\ -B}$

$\left. = 0 \right), 32 = \left(du_a = 0 \right), 33 = \left(u_{-a} = -\frac{f_{-A}}{fdot} \right), 34 = \left(\mu = (c1 - 1)\, p + c2\, \omega^2 \right), 38\ r$

$= P_{a,\ b} fdot_{-B} + \omega_{a,\ b} f_{-B} - \frac{1}{3} \theta P_{a,\ b} f_{-B}, 35 = \left(dotomega_{-a,\ -b} = -\frac{2}{3} \theta \omega_{-a,\ -b} \right), 12\ b$

$= \left(\omega^2 = \omega_a \omega_{-a} \right), 36\ a = \left(P_{a,\ -b} dotomega_b = dotomega_a \right), 11\ m1 = \left(\omega_{a,\ -b} \right.$

$\left. = \eta_{a,\ -b,\ -c,\ -d} u_d \omega_c \right), 7\ a = \left(\theta = u_{d,\ -D} \right), 10\ a = \left(\omega_b = \frac{1}{2} \eta_{b,\ e,\ f,\ g} u_{-e} \omega_{-f,\ -g} \right), 38\ l$

$= dotf_{-B} u_a u_b + dotf_A, 11\ m = \left(\omega_{-a,\ -b} = \eta_{-a,\ -b,\ -c,\ -d} u_d \omega_c \right), 11\ m2 = \left(\omega_{a,\ b} \right.$

$\left. = \eta_{a,\ b,\ -e,\ -f} u_f \omega_e \right), 12\ a = \left(\omega^2 = \frac{1}{2} \omega_{a,\ b} \omega_{-a,\ -b} \right), 36\ c = \left(dotomega = -\frac{2}{3} \theta \omega \right), 16\ b$

$= \left(\omega_{-a,\ -b} = 0 \right), 14\ b = \left(\omega_{-a,\ c} \omega_{-c,\ d} \omega_{-d,\ b} = -\omega^2 \omega_{-a,\ b} \right), 27\ a = \left(\omega_{a,\ -A} = 0 \right), 27\ b$

$= \left(\eta_{a,\ b,\ c,\ d} u_{-a} \omega_{-c,\ -d,\ -B} = 0 \right), 14\ a = \left(\omega_{-a,\ c} \omega_{-c,\ b} = -\omega^2 P_{-a,\ b} + \omega_b \omega_{-a} \right)])$

```
> PrintSubArray(eq, 1, 38, y);
```

$$1, T_{a\,b} = \rho\, u_a u_b$$

$$2, P_{a\,b} = u\, u_{a\,b} + g_{a\,b}$$

$$3, P^a{}_b u^b = 0$$

$$4, dX^a = u^b X^a{}_{;b}$$

$$5, du^a = u^b u^a{}_{;b}$$

$$6, u_{a\,;b} = \frac{1}{3} \theta P_{a\,b} + \sigma_{a\,b} + \omega_{a\,b} - du_a u_b$$

$$7,\ \theta = u^{a}{}_{;a}$$

$$8,\ \sigma_{ab} = \frac{1}{2} P_{a}{}^{c} P_{b}{}^{d} u_{c;d} + \frac{1}{2} P_{b}{}^{c} P_{a}{}^{d} u_{c;d} - \frac{1}{3} \theta P_{ab}$$

$$9,\ \omega_{ab} = \frac{1}{2} P_{a}{}^{c} P_{b}{}^{d} u_{c;d} - \frac{1}{2} P_{b}{}^{c} P_{a}{}^{d} u_{c;d}$$

$$10,\ \omega^{a} = \frac{1}{2} \eta^{abcd} u_{b} \omega_{cd}$$

$$11,\ \omega_{ab} = \eta_{abef} \omega^{e} u^{f}$$

$$12,\ \omega^{2} = \frac{1}{2} \omega^{ab} \omega_{ab}$$

13, "iff(iff(omega[-a,-b] = 0,omega[-a]),omega = 0)"

$$14,\ \omega_{a}{}^{c} \omega_{c}{}^{b} = -\omega^{2} P_{a}{}^{b} + \omega^{b} \omega_{a}$$

$$15,\ \frac{1}{2} u_{b;a} - \frac{1}{2} u_{a;b} = \frac{1}{2} du_{a} u_{b} - \frac{1}{2} du_{b} u_{a} + \omega^{ab}$$

$$16,\ -\frac{1}{6} u_{c} u_{a;b} + \frac{1}{6} u_{c} u_{b;a} + \frac{1}{6} u_{b} u_{a;c} - \frac{1}{6} u_{b} u_{c;a} - \frac{1}{6} u_{a} u_{b;c} + \frac{1}{6} u_{a} u_{c;b} = 0$$

$$17,\ \sigma_{ab} = 0$$

$$18,\ u_{a;b} = \frac{1}{3} \theta P_{ab} + \omega_{ab}$$

$$19,\ u^{a}{}_{;c;d} - u^{a}{}_{;d;c} = R^{a}{}_{bcd} u^{b}$$

$$20,\ dottheta + \frac{1}{3} \theta^{2} - 2 \omega^{2} + \frac{1}{2} \mu = 0$$

$$21,\ P_{a}{}^{c} P_{b}{}^{d} \omega_{cd;f} u^{f} + \frac{2}{3} \theta \omega_{ab} = 0$$

$$22,\ \omega_{a} \omega_{b} - \frac{1}{3} P_{ab} \omega^{2} + E_{ab} = 0$$

$$23,\ E_{ab} = C_{abcd} u^{c} u^{d}$$

$$24,\ H_{ab} = \frac{1}{2} \eta_{ae}{}^{cd} C_{cdbf} u^{e} u^{f}$$

$$25,\ P^{a}{}_{b} \omega^{b}{}_{;f} u^{f} + \frac{2}{3} \theta \omega^{a} = 0$$

$$26,\ 2 P^{ab} \theta_{;b} + 3 P^{a}{}_{b} \omega^{bd}{}_{;d}$$

$$27,\ \omega^{a}{}_{;a} = 2\, du^{a} \omega_{a}$$

$$28,\ H_{ab} = \frac{1}{2} P_{a}{}^{c} P_{b}{}^{d} \omega_{d}{}^{;c} + \frac{1}{2} P_{b}{}^{c} P_{a}{}^{d} \omega_{d}{}^{;c}$$

$$29, \omega_{a\,n}\,\omega^{n\;\;m}{}_{;m} = \omega^2\, du_a + P_a{}^b\,\omega^c\,\omega_{b\,;c} - P_a{}^b\,\omega^c\,\omega_{c\,;b} - du_p\,\omega^p\,\omega_a$$

$$30, \mu\,\theta + dotmu = 0$$

$$31, (\mu + p)\ du^a + P^{a\,b}\,p_{;b}$$

$$32, du^a = 0$$

$$33, u_a = -\frac{f_{;a}}{fdot}$$

$$34, \mu = (c1 - 1)\ p + c2\,\omega^2$$

$$35, dotomega_{a\,b} = -\frac{2}{3}\,\theta\,\omega_{a\,b}$$

$$36, dotomega = -\frac{2}{3}\,\theta\,\omega$$

$$37, \theta\left(c1p - \frac{1}{3}\,c2\,\omega^2\right) = 0$$

$$38, \frac{\partial}{\partial t}\left(P^{a\,b} f_{;b}\right) = P^{a\,b} fdot_{;b} + \omega^{a\,b} f_{;b} - \frac{1}{3}\,\theta\,P^{a\,b} f_{;b} \qquad \textbf{(1.25)}$$

>

>