```
> restart;
> with(Riemann):with(Canon):
> with(TensorPack) : CDF(0) : CDS(index) :
```

Chapter XX

Tensor analysis using indices - Senovilla et al. - Shearfree for dust page 2 if $\sigma_{a\,b}$=0 =>ωΘ=0, Author: Peter Huf, file 2c: eqs 20-25

```
> read "EFE" : read "SFE" :read "fids" :read "Seneqs2b" : read "deqs1b" :
```

the time propogation along **u** produces eqs 20, 21, 22:

****

Equation 20 Raychaudri equation

****

The "evolution" equations (with slight variations form the previous chapters) are:

```
> eq[20] := SFE[3] : T(%);
```

$$thetadot + \frac{1}{3}\,\theta^2 - 2\,\omega^2 + \frac{1}{2}\,\mu = 0 \qquad \textbf{(1.1)}$$

```
> eq[20] := u[a,-A,-E]·u[e] + (1/3)·theta·theta − 2·omega·omega + (1/2)·mu = 0 : T(%);
```

$$u^{a}{}_{;a\,;e}\,u^{e} + \frac{1}{3}\,\theta^2 - 2\,\omega^2 + \frac{1}{2}\,\mu = 0 \qquad \textbf{(1.2)}$$

```
> eq[20] := dotT(theta) + (1/3)·theta·theta − 2·omega·omega + (1/2)·mu = 0 : T(%);
```

$$dottheta + \frac{1}{3}\,\theta^2 - 2\,\omega^2 + \frac{1}{2}\,\mu = 0 \qquad \textbf{(1.3)}$$

this is an accepted proof that is common in the literature (Ellis, 1970)

****

Equation 21

****

```
> eq[21] := P[-a, c]·P[-b, d]·omega[-c,-d,-F]·u[f] + (2/3)·theta·omega[-a,-b] = 0 :
    T(%);
```

$$P_{a}{}^{c}\,P_{b}{}^{d}\,\omega_{c\,d\,;f}\,u^{f} + \frac{2}{3}\,\theta\,\omega_{a\,b} = 0 \qquad \textbf{(1.4)}$$

this is an accepted proof in the literature (Ellis, 1970)

****

Equation 22

****

Equation 22 is the field equation 5, with the dust assumption.

```
> T(SFE[5]);
```

(1.5)

$$\omega_a \omega_b - \frac{1}{3} P_{ab} \omega^2 + E_{ab} = 0 \quad (1.5)$$

```
> eq[22] := expand(TEDS(du[-g,-F]=0, SFE[5])) : T(%);
```

$$\omega_a \omega_b - \frac{1}{3} P_{ab} \omega^2 + E_{ab} = 0 \quad (1.6)$$

****

Equations 23 & 24 - definitions of E_{ab} and H_{ab}

****

The definitions of E and H (eqs 23 and 24):

```
> eq[23] := E[-a,-b]=C[-a,-b,-c,-d]·u[c]·u[d] : T(%);
```

$$E_{ab} = C_{abcd} u^c u^d \quad (1.7)$$

```
> eq[24] := H[-a,-b] = 1/2 ·eta[-a,-e, c, d]·C[-c,-d,-b,-f]·u[e]·u[f] : T(%);
```

$$H_{ab} = \frac{1}{2} \eta_{ae}{}^{cd} C_{cdbf} u^e u^f \quad (1.8)$$

****

Equation 25

****

```
> eq[11] : T(%);
```

$$\omega_{ab} = \eta_{abef} \omega^e u^f \quad (1.9)$$

```
> temp1 := subs(a=c, b=d, eq[11]) : T(%);
```

$$\omega_{cd} = \eta_{cdef} \omega^e u^f \quad (1.10)$$

```
> temp2 := cod(temp1,-h) : T(%);
```

$$\omega_{cd;h} = \eta_{cdef} \omega^e u^f{}_{;h} + \eta_{cdef} \omega^e{}_{;h} u^f + \eta_{cdef;h} \omega^e u^f \quad (1.11)$$

We start with eq21:

```
> eq[21] : T(%);
```

$$P_a{}^c P_b{}^d \omega_{cd;f} u^f + \frac{2}{3} \theta \omega_{ab} = 0 \quad (1.12)$$

Term 1 of eq21 has a time derivative component : $\omega_{cd;h} u^h = \left(\eta_{cdef} \omega^e u^f\right)_{;h} u^h$

The RHS = $\left(\eta_{cdef} \omega^e{}_{;h} u^f\right) u^h + \left(\eta_{cdef} \omega^e u^f{}_{;h}\right) u^h$

```
> u[e,-H] : T(%);
```

$$u^e{}_{;h} \quad (1.13)$$

```
> temp3 := expand(TEDS(eq[11], subs(f=h, F=H, eq[21]))) : T(%);
```

$$P_a{}^c P_b{}^d \omega_{cd;h} u^h + \frac{2}{3} \theta \eta_{abef} \omega^e u^f = 0 \quad (1.14)$$

> *temp4* := *expand*(*TEDS*(*temp2*, *temp3*)) : *T*(%);

$$P_{a}{}^{c}P_{b}{}^{d}\eta_{cdef}\omega^{e}u^{h}u^{f}{}_{;h}+P_{a}{}^{c}P_{b}{}^{d}\eta_{cdef}\omega^{e}{}_{;h}u^{f}u^{h}+P_{a}{}^{c}P_{b}{}^{d}\eta_{cdef;h}\omega^{e}u^{f}u^{h}+\frac{2}{3}\theta\eta_{abef}\omega^{e}u^{f}=0 \quad (1.15)$$

> *temp5* := *expand*(*TEDS*(eta[−*c*, −*d*, −*e*, −*f*, −*H*] = 0, *temp4*)) : *T*(%);

$$P_{a}{}^{c}P_{b}{}^{d}\eta_{cdef}\omega^{e}u^{h}u^{f}{}_{;h}+P_{a}{}^{c}P_{b}{}^{d}\eta_{cdef}\omega^{e}{}_{;h}u^{f}u^{h}+\frac{2}{3}\theta\eta_{abef}\omega^{e}u^{f}=0 \quad (1.16)$$

>

> *subs*(*b* = *c*, *deq*[1]) : *T*(%);

$$P_{ac}=u\,u_{ac}+g_{ac} \quad (1.17)$$

> *temp6* := *expand*(*TEDS*(*subs*(*b* = −*c*, *deq*[1]), *temp5*)) : *T*(%);

$$P_{b}{}^{d}\eta_{cdef}\omega^{e}u^{c}u^{h}u_{a}u^{f}{}_{;h}+P_{b}{}^{d}\eta_{cdef}g_{a}{}^{c}\omega^{e}u^{h}u^{f}{}_{;h}+P_{b}{}^{d}\eta_{cdef}\omega^{e}{}_{;h}u^{c}u^{f}u^{h}u_{a}+P_{b}{}^{d}\eta_{cdef}g_{a}{}^{c}\omega^{e}{}_{;h}u^{f}u^{h}+\frac{2}{3}\theta\eta_{abef}\omega^{e}u^{f}=0 \quad (1.18)$$

> *temp7* := *expand*(*TEDS*(*subs*(*b* = −*d*, *a* = *b*, *deq*[1]), *temp6*)) : *T*(%);

$$\eta_{cdef}\omega^{e}u^{c}u^{d}u^{h}u_{a}u_{b}u^{f}{}_{;h}+\eta_{cdef}g_{b}{}^{d}\omega^{e}u^{c}u^{h}u_{a}u^{f}{}_{;h}+\eta_{cdef}g_{a}{}^{c}\omega^{e}u^{d}u^{h}u_{b}u^{f}{}_{;h}+\eta_{cdef}g_{a}{}^{c}g_{b}{}^{d}\omega^{e}u^{h}u^{f}{}_{;h}+\eta_{cdef}\omega^{e}{}_{;h}u^{c}u^{d}u^{f}u^{h}u_{a}u_{b}+\eta_{cdef}g_{b}{}^{d}\omega^{e}{}_{;h}u^{c}u^{f}u^{h}u_{a}+\eta_{cdef}g_{a}{}^{c}\omega^{e}{}_{;h}u^{d}u^{f}u^{h}u_{b}+\eta_{cdef}g_{a}{}^{c}g_{b}{}^{d}\omega^{e}{}_{;h}u^{f}u^{h}+\frac{2}{3}\theta\eta_{abef}\omega^{e}u^{f}=0 \quad (1.19)$$

> *temp8* := *Absorbg*(*Absorbg*(*temp7*)) : *T*(%);

0, "not a tensor"

0, "not a tensor"

$$\eta_{cdef}\omega^{e}u^{c}u^{d}u^{h}u_{a}u_{b}u^{f}{}_{;h}+\eta_{cbef}\omega^{e}u^{c}u^{h}u_{a}u^{f}{}_{;h}+\eta_{adef}\omega^{e}u^{d}u^{h}u_{b}u^{f}{}_{;h}+\eta_{abef}\omega^{e}u^{h}u^{f}{}_{;h}+\eta_{cdef}\omega^{e}{}_{;h}u^{c}u^{d}u^{f}u^{h}u_{a}u_{b}+\eta_{cbef}\omega^{e}{}_{;h}u^{c}u^{f}u^{h}u_{a}+\eta_{adef}\omega^{e}{}_{;h}u^{d}u^{f}u^{h}u_{b}+\eta_{abef}\omega^{e}{}_{;h}u^{f}u^{h}+\frac{2}{3}\theta\eta_{abef}\omega^{e}u^{f}=0 \quad (1.20)$$

> *temp9* := *expand*(*TEDS*(*u*[*f*, −*H*]·*u*[*h*] = 0, *temp8*)) : *T*(%);

$$\eta_{cdef}\omega^{e}{}_{;h}u^{c}u^{d}u^{f}u^{h}u_{a}u_{b}+\eta_{cbef}\omega^{e}{}_{;h}u^{c}u^{f}u^{h}u_{a} \quad (1.21)$$

$$+\eta_{adef}\omega^{e}{}_{;h}u^{d}u^{f}u^{h}u_{b}+\eta_{abef}\omega^{e}{}_{;h}u^{f}u^{h}+\frac{2}{3}\theta\,\eta_{abef}\omega^{e}u^{f}=0$$

There are several terms involving symmetric and antisymmetric tensors that vanish:

```
> temp10 := expand(TEDS(eta[-c,-d,-e,-f]·u[f]·u[d]=0, temp9)) : T(%);
```

$$\eta_{cbef}\omega^{e}{}_{;h}u^{c}u^{f}u^{h}u_{a}+\eta_{adef}\omega^{e}{}_{;h}u^{d}u^{f}u^{h}u_{b}+\eta_{abef}\omega^{e}{}_{;h}u^{f}u^{h}+\frac{2}{3}\theta\,\eta_{abef}\omega^{e}u^{f}=0 \quad (1.22)$$

```
> temp11 := expand(TEDS(eta[-a,-d,-e,-f]·u[f]·u[d]=0, temp10)) : T(%);
```

$$\eta_{cbef}\omega^{e}{}_{;h}u^{c}u^{f}u^{h}u_{a}+\eta_{abef}\omega^{e}{}_{;h}u^{f}u^{h}+\frac{2}{3}\theta\,\eta_{abef}\omega^{e}u^{f}=0 \quad (1.23)$$

```
> temp12 := expand(TEDS(eta[-c,-b,-e,-f]·u[f]·u[c]=0, temp11)) : T(%);
```

$$\eta_{abef}\omega^{e}{}_{;h}u^{f}u^{h}+\frac{2}{3}\theta\,\eta_{abef}\omega^{e}u^{f}=0 \quad (1.24)$$

```
> temp13 := factor(temp12) : T(%);
```

$$\frac{1}{3}\eta_{abef}u^{f}\left(2\theta\omega^{e}+3\omega^{e}{}_{;h}u^{h}\right)=0 \quad (1.25)$$

assuming u<>0 then the bracket term above must be zero (since we assume velocity is not zero):

```
> temp14 := 1/3 · op(4, op(1, temp13)) = 0 : T(%);
```

$$\frac{2}{3}\theta\omega^{e}+\omega^{e}{}_{;h}u^{h}=0 \quad (1.26)$$

```
> temp15 := expand(TEDS(omega[e,-H]·u[h]=dotomega[e], temp14)) : T(%);
```

$$\frac{2}{3}\theta\omega^{e}+dotomega^{e}=0 \quad (1.27)$$

```
>
```

now we compare to eq25

```
> eq[25] := P[a,-b]·omega[b,-F]·u[f] + (2/3)·theta·omega[a]=0 : T(%);
```

$$P^{a}{}_{b}\omega^{b}{}_{;f}u^{f}+\frac{2}{3}\theta\omega^{a}=0 \quad (1.28)$$

```
> subs(b=b, a=-a, deq[1]) : T(%);
```

$$P^{a}{}_{b}=u^{a}u_{b}+g^{a}{}_{b} \quad (1.29)$$

```
> proof[25] := expand(TEDS(subs(b=b, a=-a, deq[1]), eq[25])) : T(%);
```

$$\omega^{b}{}_{;f}u^{a}u^{f}u_{b}+g^{a}{}_{b}\omega^{b}{}_{;f}u^{f}+\frac{2}{3}\theta\omega^{a}=0 \quad (1.30)$$

```
> proof[25] := Absorbg(proof[25]) : T(%);
```

0, "not a tensor"

(1.31)

$$\omega^{b}{}_{;f}u^{a}u^{f}u_{b}+\omega^{a}{}_{;f}u^{f}+\frac{2}{3}\theta\omega^{a}=0 \quad \textbf{(1.31)}$$

> *proof*[25] := *expand*(*TEDS*(omega[*a*,-*F*]·*u*[*f*] = *dotomega*[*a*], *proof*[25])) : *T*(%);

$$\omega^{b}{}_{;f}u^{a}u^{f}u_{b}+dotomega^{a}+\frac{2}{3}\theta\omega^{a}=0 \quad \textbf{(1.32)}$$

> *proof*[25] := *expand*(*TEDS*(omega[*b*,-*F*]·*u*[*f*] = *dotomega*[*b*], *proof*[25])) : *T*(%);

$$u^{a}u_{b}dotomega^{b}+dotomega^{a}+\frac{2}{3}\theta\omega^{a}=0 \quad \textbf{(1.33)}$$

now

> *proof*[25] := *expand*(*TEDS*(*dotomega*[*b*]·*u*[-*b*] = 0, *proof*[25])) : *T*(%);

$$dotomega^{a}+\frac{2}{3}\theta\omega^{a}=0 \quad \textbf{(1.34)}$$

which is equivalent to temp15

> *temp15* : *T*(%);

$$\frac{2}{3}\theta\omega^{e}+dotomega^{e}=0 \quad \textbf{(1.35)}$$

> *temp15* : *T*(%);

$$\frac{2}{3}\theta\omega^{e}+dotomega^{e}=0 \quad \textbf{(1.36)}$$

>

>

eq25 is proven***

> **save** *eq*, "Seneqs2c" :

go to page 3

> **read** "Seneqs2c";

$$eq := table\Big(\Big[1=\big(TensorPack{:}\text{-}T_{-a,\,-b}=\rho\,u_{-a}u_{-b}\big),2=\big(P_{-a,\,-b}=u_{-a}u_{-b}+g_{-a,\,-b}\big),3=\big(P_{a,\,-b}u_{b}=0\big),4=\big(dX_{a}=u_{b}X_{a,\,-B}\big),5=\big(du_{a}=u_{b}u_{a,\,-B}\big),6=\Big(u_{-a,\,-B}=\frac{1}{3}\theta P_{-a,\,-b}+\sigma_{-a,\,-b}+\omega_{-a,\,-b}-du_{-a}u_{-b}\Big),7\,a=\big(\theta=u_{d,\,-D}\big),7=\big(\theta=u_{a,\,-A}\big),9=\Big(\omega_{-a,\,-b}=\frac{1}{2}P_{-a,\,c}P_{-b,\,d}u_{-c,\,-D}-\frac{1}{2}P_{-b,\,c}P_{-a,\,d}u_{-c,\,-D}\Big),8=\Big(\sigma_{-a,\,-b}=\frac{1}{2}P_{-a,\,c}P_{-b,\,d}u_{-c,\,-D}+\frac{1}{2}P_{-b,\,c}P_{-a,\,d}u_{-c,\,-D}-\frac{1}{3}\theta P_{-a,\,-b}\Big),11=\big(\omega_{-a,\,-b}=\eta_{-a,\,-b,\,-e,\,-f}\omega_{e}u_{f}\big),10=\Big(\omega_{a} \quad \textbf{(1.37)}$$

$= \frac{1}{2}\eta_{a,b,c,d}u_{-b}\omega_{-c,-d}\Big)$, 13 = "iff(iff(omega[-a,-b] = 0,omega[-a]),omega = 0)", 12 $= \Big(\omega^2 = \frac{1}{2}\omega_{a,b}\omega_{-a,-b}\Big)$, $15 = \Big(\frac{1}{2}u_{-b,-A} - \frac{1}{2}u_{-a,-B} = \frac{1}{2}du_{-a}u_{-b} - \frac{1}{2}du_{-b}u_{-a} + \omega_{a,b}\Big)$, $14 = \big(\omega_{-a,c}\omega_{-c,b} = -\omega^2 P_{-a,b} + \omega_b\omega_{-a}\big)$, $18 = \Big(u_{-b,-A} = -u_{-a}u_{-b,-C}u_c + \frac{1}{3}\theta h_{-a,-b} + \omega_{-a,-b}\Big)$, $19 = \big(u_{a,-C,-D} - u_{a,-D,-C} = R_{a,-b,-c,-d}u_b\big)$, $16 = \Big(-\frac{1}{6}u_{-c}u_{-a,-B} + \frac{1}{6}u_{-c}u_{-b,-A} + \frac{1}{6}u_{-b}u_{-a,-C} - \frac{1}{6}u_{-b}u_{-c,-A} - \frac{1}{6}u_{-a}u_{-b,-C} + \frac{1}{6}u_{-a}u_{-c,-B} = 0\Big)$, $17 = \big(\sigma_{-a,-b} = 0\big)$, $22 = \Big(\omega_{-a}\omega_{-b} - \frac{1}{3}P_{-a,-b}\omega^2 + E_{-a,-b} = 0\Big)$, $23 = \big(E_{-a,-b} = C_{-a,-b,-c,-d}u_c u_d\big)$, $20 = \Big(dottheta + \frac{1}{3}\theta^2 - 2\omega^2 + \frac{1}{2}\mu = 0\Big)$, $21 = \Big(P_{-a,c}P_{-b,d}\omega_{-c,-d,-F}u_f + \frac{2}{3}\theta\omega_{-a,-b} = 0\Big)$, $25 = \Big(P_{a,-b}\omega_{b,-F}u_f + \frac{2}{3}\theta\omega_a = 0\Big)$, $14\,a = \big(\omega_{-a,c}\omega_{-c,b} = -\omega^2 P_{-a,b} + \omega_b\omega_{-a}\big)$, $24 = \Big(H_{-a,-b} = \frac{1}{2}\eta_{-a,-e,c,d}C_{-c,-d,-b,-f}u_e u_f\Big)$, $10\,a = \Big(\omega_b = \frac{1}{2}\eta_{b,e,f,g}u_{-e}\omega_{-f,-g}\Big)$, $12\,b = \big(\omega^2 = \omega_a\omega_{-a}\big)$, $16\,a = \Big(-\frac{1}{6}u_{-c}u_{-a,-B} + \frac{1}{6}u_{-c}u_{-b,-A} + \frac{1}{6}u_{-b}u_{-a,-C} - \frac{1}{6}u_{-b}u_{-c,-A} - \frac{1}{6}u_{-a}u_{-b,-C} + \frac{1}{6}u_{-a}u_{-c,-B} = 0\Big)$, $12\,a = \Big(\omega^2 = \frac{1}{2}\omega_{a,b}\omega_{-a,-b}\Big)$, $14\,b = \big(\omega_{-a,c}\omega_{-c,d}\omega_{-d,b} = -\omega^2\omega_{-a,b}\big)$, $11\,m = \big(\omega_{-a,-b} = \eta_{-a,-b,-c,-d}u_d\omega_c\big)$, $11\,m2 = \big(\omega_{a,b} = \eta_{a,b,-e,-f}u_f\omega_e\big)$, $16\,b = \big(\omega_{-a,-b} = 0\big)$, $10\,b = \Big(\eta_{-f,-g,-a,-e}\omega_a u_e = \frac{1}{2}\eta_{-f,-g,-a,-e}\eta_{a,b,c,d}u_{-b}\omega_{-c,-d}u_e\Big)$, $11\,m1 = \big(\omega_{a,-b} = \eta_{a,-b,-c,-d}u_d\omega_c\big)\big]\big)$

> PrintSubArray(eq, 1, 25, y);

$$1,\ T_{a\,b} = \rho\, u_a u_b$$

$$2,\ P_{a\,b} = u\, u_{b\,a} + g_{a\,b}$$

$$3,\ P^{a}{}_{b}\, u^{b} = 0$$

$$4,\ dX^{a} = u^{b} X^{a}{}_{;b}$$

$$5,\ du^{a} = u^{b} u^{a}{}_{;b}$$

$$6,\ u_{a\,;b} = \frac{1}{3}\theta P_{a\,b} + \sigma_{a\,b} + \omega_{a\,b} - du_a u_b$$

$$7, \theta = u^{a}{}_{;a}$$

$$8, \sigma_{ab} = \frac{1}{2} P_{a}{}^{c} P_{b}{}^{d} u_{c;d} + \frac{1}{2} P_{b}{}^{c} P_{a}{}^{d} u_{c;d} - \frac{1}{3} \theta P_{ab}$$

$$9, \omega_{ab} = \frac{1}{2} P_{a}{}^{c} P_{b}{}^{d} u_{c;d} - \frac{1}{2} P_{b}{}^{c} P_{a}{}^{d} u_{c;d}$$

$$10, \omega^{a} = \frac{1}{2} \eta^{abcd} u_{b} \omega_{cd}$$

$$11, \omega_{ab} = \eta_{abef} \omega^{e} u^{f}$$

$$12, \omega^{2} = \frac{1}{2} \omega^{ab} \omega_{ab}$$

13, "iff(iff(omega[-a,-b] = 0,omega[-a]),omega = 0)"

$$14, \omega_{a}{}^{c} \omega_{c}{}^{b} = -\omega^{2} P_{a}{}^{b} + \omega^{b} \omega_{a}$$

$$15, \frac{1}{2} u_{b;a} - \frac{1}{2} u_{a;b} = \frac{1}{2} du_{a} u_{b} - \frac{1}{2} du_{b} u_{a} + \omega^{ab}$$

$$16, -\frac{1}{6} u_{c} u_{a;b} + \frac{1}{6} u_{c} u_{b;a} + \frac{1}{6} u_{b} u_{a;c} - \frac{1}{6} u_{b} u_{c;a} - \frac{1}{6} u_{a} u_{b;c} + \frac{1}{6} u_{a} u_{c;b} = 0$$

$$17, \sigma_{ab} = 0$$

$$18, u_{b;a} = -u_{a} u_{b;c} u^{c} + \frac{1}{3} \theta h_{ab} + \omega_{ab}$$

$$19, u^{a}{}_{;c;d} - u^{a}{}_{;d;c} = R^{a}{}_{bcd} u^{b}$$

$$20, dottheta + \frac{1}{3} \theta^{2} - 2\omega^{2} + \frac{1}{2} \mu = 0$$

$$21, P_{a}{}^{c} P_{b}{}^{d} \omega_{cd;f} u^{f} + \frac{2}{3} \theta \omega_{ab} = 0$$

$$22, \omega_{a} \omega_{b} - \frac{1}{3} P_{ab} \omega^{2} + E_{ab} = 0$$

$$23, E_{ab} = C_{abcd} u^{c} u^{d}$$

$$24, H_{ab} = \frac{1}{2} \eta_{ae}{}^{cd} C_{cdbf} u^{e} u^{f}$$

$$25, P^{a}{}_{b} \omega^{b}{}_{;f} u^{f} + \frac{2}{3} \theta \omega^{a} = 0 \qquad \textbf{(1.38)}$$

>

>