> **restart;**

> **with(Riemann):with(Canon):**

> *with*(*TensorPack*) : *CDF*(0) : *CDS*(*index*) :

Chapter XX
Tensor analysis using indices - Senovilla et al. - Shearfree for dust
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if $\sigma_{a\,b}$=0 =>ωΘ=0
Author: Peter Huf
file 2b: eqs 15-19

> **read** "EFE" : **read** "SFE" :**read** "fids" :**read** "Seneqs2a" :

****
Equation 15 - relation between the antisymmetric omponents of the covariant derivative of velocity, acceleration and vorticity
****

The following is an important identity that follows from the antisymmetry of eq 6:

> *eq*[15] := *antisymm*(*u*[−*b*,−*A*],−*b*,−*A*) = *antisymm*(*du*[−*a*]·*u*[−*b*],−*a*,−*b*) + omega[*a*, *b*] : *T*(*eq*[15]);

$$\frac{1}{2}\,u_{b\,;a}-\frac{1}{2}\,u_{a\,;b}=\frac{1}{2}\,du_{a}\,u_{b}-\frac{1}{2}\,du_{b}\,u_{a}+\omega^{a\;b} \quad \textbf{(1.1)}$$

> *eq*[16 *a*] := *antisymm*(*u*[−*c*]·*u*[−*b*,−*A*],−*c*,−*A*) = 0 : *T*(*eq*[15]);

$$\frac{1}{2}\,u_{b\,;a}-\frac{1}{2}\,u_{a\,;b}=\frac{1}{2}\,du_{a}\,u_{b}-\frac{1}{2}\,du_{b}\,u_{a}+\omega^{a\;b} \quad \textbf{(1.2)}$$

> *eq*[16 *b*] := omega[−*a*,−*b*] = 0 : *T*(%);

$$\omega_{a\,b}=0 \quad \textbf{(1.3)}$$

if eq 15a <=> eq15b

or, written in text:
$u_{[c}u_{b\,;\,a]}$=0 <=> $\omega_{a\,b}=0$

Proof: . This equation is used by Senovilla, but not in our current proof.

The full version of eq 6,

> *eq*[6] := *u*[−*a*,−*B*] = $\left(\frac{1}{3}\right)$· theta·*P*[−*a*,−*b*] + sigma[−*a*,−*b*] + omega[−*a*,−*b*] − *du*[−*a*] ·*u*[−*b*] : *T*(%);

(1.4)

$$u_{a;b} = \frac{1}{3}\theta P_{ab} + \sigma_{ab} + \omega_{ab} - du_a u_b \quad (1.4)$$

reduces with σ=0 to:

> proof[15] := u[-a,-B] = (1/3) · theta·P[-a,-b] + omega[-a,-b] − du[-a]·u[-b] :
> T(%);

$$u_{a;b} = \frac{1}{3}\theta P_{ab} + \omega_{ab} - du_a u_b \quad (1.5)$$

The antisymmetric component is

> proof[15] := antisymm(u[-a,-B],-a,-B) = expand((1/3) · theta·antisymm(P[-a,-b],-a,
> -b) + omega[-a,-b] − antisymm(du[-a]·u[-b],-a,-b)) : T(%);

$$\frac{1}{2}u_{a;b} - \frac{1}{2}u_{b;a} = \frac{1}{6}\theta P_{ab} - \frac{1}{6}\theta P_{ba} + \omega_{ab} - \frac{1}{2}du_a u_b + \frac{1}{2}du_b u_a \quad (1.6)$$

since P_{ab} is symmetrical:

> proof[15] := expand(TEDS(P[-a,-b] = P[-b,-a], proof[15])) : T(%);

$$\frac{1}{2}u_{a;b} - \frac{1}{2}u_{b;a} = \omega_{ab} - \frac{1}{2}du_a u_b + \frac{1}{2}du_b u_a \quad (1.7)$$

which is eq15.

****

Equation 16 - extension of eq15

****

> eq[16] := antisymm(u[-c]·u[-b,-A],-c,-A) = 0 : T(%);

$$-\frac{1}{6}u_c u_{a;b} + \frac{1}{6}u_c u_{b;a} + \frac{1}{6}u_b u_{a;c} - \frac{1}{6}u_b u_{c;a} - \frac{1}{6}u_a u_{b;c} + \frac{1}{6}u_a u_{c;b} = 0 \quad (1.8)$$

Now

> proof[16] := antisymm(u[-c]·u[-b,-A],-c,-A) = 0 : T(%);

$$-\frac{1}{6}u_c u_{a;b} + \frac{1}{6}u_c u_{b;a} + \frac{1}{6}u_b u_{a;c} - \frac{1}{6}u_b u_{c;a} - \frac{1}{6}u_a u_{b;c} + \frac{1}{6}u_a u_{c;b} = 0 \quad (1.9)$$

and

> (1/3) ·proof[15] : T(%);

$$\frac{1}{6}u_{a;b} - \frac{1}{6}u_{b;a} = \frac{1}{3}\omega_{ab} - \frac{1}{6}du_a u_b + \frac{1}{6}du_b u_a \quad (1.10)$$

> proof[16] := expand(TEDS((1/3) ·proof[15], proof[16])) : T(%);

$$-\frac{1}{3}u_c\omega_{ab} + \frac{1}{6}u_c du_a u_b - \frac{1}{6}u_c du_b u_a + \frac{1}{6}u_b u_{a;c} - \frac{1}{6}u_b u_{c;a} - \frac{1}{6}u_a u_{b;c} + \frac{1}{6}u_a u_{c;b} = 0 \quad (1.11)$$

> subs$\left(-a=-d,-A=-\mathrm{D},-b=-c,-B=-C,-d=-b,-\mathrm{D}=-B,\ \frac{1}{3}\cdot proof[15]\right)$: T(%);

$$\frac{1}{6}\,u_{b\,;c}-\frac{1}{6}\,u_{c\,;b}=\frac{1}{3}\,\omega_{b\,c}-\frac{1}{6}\,du_{b}\,u_{c}+\frac{1}{6}\,du_{c}\,u_{b} \qquad (1.12)$$

> proof[16] := expand$\left(TEDS\left(subs\left(-a=-d,-A=-\mathrm{D},-b=-c,-B=-C,-d=-b,-\mathrm{D}=-B,\ \frac{1}{3}\cdot proof[15]\right), proof[16]\right)\right)$: T(%);

$$-\frac{1}{3}\,u_{c}\,\omega_{a\,b}+\frac{1}{6}\,u_{c}\,du_{a}\,u_{b}+\frac{1}{6}\,u_{b}\,u_{a\,;c}-\frac{1}{6}\,u_{b}\,u_{c\,;a}-\frac{1}{3}\,u_{a}\,\omega_{b\,c}-\frac{1}{6}\,u_{a}\,du_{c}\,u_{b}=0 \qquad (1.13)$$

> subs$\left(-b=-c,-B=-C,\ \frac{1}{3}\cdot proof[15]\right)$: T(%);

$$\frac{1}{6}\,u_{a\,;c}-\frac{1}{6}\,u_{c\,;a}=\frac{1}{3}\,\omega_{a\,c}-\frac{1}{6}\,du_{a}\,u_{c}+\frac{1}{6}\,du_{c}\,u_{a} \qquad (1.14)$$

> proof[16] := expand$\left(TEDS\left(subs\left(-b=-c,-B=-C,\ \frac{1}{3}\cdot proof[15]\right), proof[16]\right)\right)$: T(%);

$$-\frac{1}{3}\,u_{c}\,\omega_{a\,b}+\frac{1}{3}\,u_{b}\,\omega_{a\,c}-\frac{1}{3}\,u_{a}\,\omega_{b\,c}=0 \qquad (1.15)$$

> proof[16] := expand(u[a]·proof[16]) : T(%);

$$-\frac{1}{3}\,u^{a}\,u_{c}\,\omega_{a\,b}+\frac{1}{3}\,u^{a}\,u_{b}\,\omega_{a\,c}-\frac{1}{3}\,u^{a}\,u_{a}\,\omega_{b\,c}=0 \qquad (1.16)$$

> proof[16] := expand(TEDS(id[1], proof[16])) : T(%);

$$-\frac{1}{3}\,u^{a}\,u_{c}\,\omega_{a\,b}+\frac{1}{3}\,u^{a}\,u_{b}\,\omega_{a\,c}+\frac{1}{3}\,\omega_{b\,c}=0 \qquad (1.17)$$

> proof[16] := expand(TEDS(omega[-a,-b]·u[a]=0, proof[16])) : T(%);

$$\frac{1}{3}\,u^{a}\,u_{b}\,\omega_{a\,c}+\frac{1}{3}\,\omega_{b\,c}=0 \qquad (1.18)$$

> proof[16] := expand(TEDS(omega[-a,-c]·u[a]=0, proof[16])) : T(%);

$$\frac{1}{3}\,\omega_{b\,c}=0 \qquad (1.19)$$

thus proving eqs 15 and 16***

>

****

Equation Frobenius thearem

****

According to Frobenious therem, there exist locally functions f and h such that
u_{a}=h$f_{;a}$
if and only if the rotation vanishes

****

Equation 17 absence of shear

****

```
> eq[17] := sigma[-a,-b]=0 : T(%);
```

$$\sigma_{a\,b} = 0 \tag{1.20}$$

This assumption is made for the remaining paper.

****

Equation 18 Identity of eq6

****

We commence with the identity:

```
> EFE[1] : T(%);
```

$$u_{b\,;a} = -u_{a}\, u_{b\,;c}\, u^{c} + \frac{1}{3}\,\theta\, h_{a\,b} + \sigma_{a\,b} + \omega_{a\,b} \tag{1.21}$$

which reduces to (with the dust assumptions):

```
> eq[18] := SFE[1] : T(%);
```

$$u_{b\,;a} = -u_{a}\, u_{b\,;c}\, u^{c} + \frac{1}{3}\,\theta\, h_{a\,b} + \omega_{a\,b} \tag{1.22}$$

****

Equation 19 Ricci identities for velocity

****

Using the Ricci identities for velocity:

```
> eq[19] := u[a,-C,-D] - u[a,-D,-C] = R[a,-b,-c,-d]·u[b] : T(%);
```

$$u^{a}{}_{;c\,;d} - u^{a}{}_{;d\,;c} = R^{a}{}_{b\,c\,d}\, u^{b} \tag{1.23}$$

the time propogation along **u** produces eqs 20, 21, 22 - see next file.

```
> save eq, "Seneqs2b" :
```

go to page 3

```
> read "Seneqs2b";
```

$$eq := table\Big(\Big[14\,a = \big(\omega_{-a,\,c}\,\omega_{-c,\,b} = -\omega^{2}\,P_{-a,\,b} + \omega_{b}\,\omega_{-a}\big),\ 1 = \big(TensorPack\text{:-}T_{-a,\,-b} = \rho\, u_{-a}\, u_{-b}\big),\ 12\,b = \big(\omega^{2} = \omega_{a}\,\omega_{-a}\big),\ 2 = \big(P_{-a,\,-b} = u_{-a}\, u_{-b} + g_{-a,\,-b}\big),\ 3 = \big(P_{a,\,-b}\, u_{b} = 0\big),\ 4 = \big(dX_{a} = u_{b}\, X_{a,\,-B}\big),\ 5 = \big(du_{a} = u_{b}\, u_{a,\,-B}\big),\ 6 = \Big(u_{-a,\,-B} = \frac{1}{3}\,\theta\, P_{-a,\,-b} + \sigma_{-a,\,-b} + \omega_{-a,\,-b} - du_{-a}\, u_{-b}\Big),\ 7 = \big(\theta = u_{a,\,-A}\big),\ 9 = \Big(\omega_{-a,\,-b} = \frac{1}{2}\, P_{-a,\,c}\, P_{-b,\,d}\, u_{-c,\,-D} - \frac{1}{2}\, P_{-b,\,c}\, P_{-a,\,d}\, u_{-c,\,-D}\Big),\ 8 = \Big(\sigma_{-a,\,-b} = \frac{1}{2}\, P_{-a,\,c}\, P_{-b,\,d}\, u_{-c,\,-D} + \frac{1}{2}\, P_{-b,\,c}\, P_{-a,\,d}\, u_{-c,\,-D} - \frac{1}{3}\,\theta\, P_{-a,\,-b}\Big),\ 11 = \big(\omega_{-a,\,-b} = \eta_{-a,\,-b,\,-e,\,-f}\,\omega_{e}\, u_{f}\big),\ 10 = \big(\omega_{a} \tag{1.24}$$

$= \frac{1}{2}\,\eta_{a,\,b,\,c,\,d}\,u_{-b}\,\omega_{-c,\,-d}\Big)$, 13 = "iff(iff(omega[-a,-b] = 0,omega[-a]),omega = 0)", $12 = \Big(\omega^2 = \frac{1}{2}\,\omega_{a,\,b}\,\omega_{-a,\,-b}\Big)$, $15 = \Big(\frac{1}{2}\,u_{-b,\,-A} - \frac{1}{2}\,u_{-a,\,-B} = \frac{1}{2}\,du_{-a}\,u_{-b} - \frac{1}{2}\,du_{-b}\,u_{-a} + \omega_{a,\,b}\Big)$, $14 = \Big(\omega_{-a,\,c}\,\omega_{-c,\,b} = -\omega^2\,P_{-a,\,b} + \omega_b\,\omega_{-a}\Big)$, $18 = \Big(u_{-b,\,-A} = -u_{-a}\,u_{-b,\,-C}\,u_c + \frac{1}{3}\,\theta\,h_{-a,\,-b} + \omega_{-a,\,-b}\Big)$, $19 = \big(u_{a,\,-C,\,-D} - u_{a,\,-D,\,-C} = R_{a,\,-b,\,-c,\,-d}\,u_b\big)$, $16 = \Big(-\frac{1}{6}\,u_{-c}\,u_{-a,\,-B} + \frac{1}{6}\,u_{-c}\,u_{-b,\,-A} + \frac{1}{6}\,u_{-b}\,u_{-a,\,-C} - \frac{1}{6}\,u_{-b}\,u_{-c,\,-A} - \frac{1}{6}\,u_{-a}\,u_{-b,\,-C} + \frac{1}{6}\,u_{-a}\,u_{-c,\,-B} = 0\Big)$, $17 = \big(\sigma_{-a,\,-b} = 0\big)$, $11\,m = \big(\omega_{-a,\,-b} = \eta_{-a,\,-b,\,-c,\,-d}\,u_d\,\omega_c\big)$, $16\,b = \big(\omega_{-a,\,-b} = 0\big)$, $11\,m2 = \big(\omega_{a,\,b} = \eta_{a,\,b,\,-e,\,-f}\,u_f\,\omega_e\big)$, $11\,m1 = \big(\omega_{a,\,-b} = \eta_{a,\,-b,\,-c,\,-d}\,u_d\,\omega_c\big)$, $16\,a = \Big(-\frac{1}{6}\,u_{-c}\,u_{-a,\,-B} + \frac{1}{6}\,u_{-c}\,u_{-b,\,-A} + \frac{1}{6}\,u_{-b}\,u_{-a,\,-C} - \frac{1}{6}\,u_{-b}\,u_{-c,\,-A} - \frac{1}{6}\,u_{-a}\,u_{-b,\,-C} + \frac{1}{6}\,u_{-a}\,u_{-c,\,-B} = 0\Big)$, $7\,a = \big(\theta = u_{d,\,-D}\big)$, $12\,a = \Big(\omega^2 = \frac{1}{2}\,\omega_{a,\,b}\,\omega_{-a,\,-b}\Big)$, $10\,b = \Big(\eta_{-f,\,-g,\,-a,\,-e}\,\omega_a\,u_e = \frac{1}{2}\,\eta_{-f,\,-g,\,-a,\,-e}\,\eta_{a,\,b,\,c,\,d}\,u_{-b}\,\omega_{-c,\,-d}\,u_e\Big)$, $10\,a = \Big(\omega_b = \frac{1}{2}\,\eta_{b,\,e,\,f,\,g}\,u_{-e}\,\omega_{-f,\,-g}\Big)$, $14\,b = \Big(\omega_{-a,\,c}\,\omega_{-c,\,d}\,\omega_{-d,\,b} = -\omega^2\,\omega_{-a,\,b}\Big)\Big]\Big)$

> PrintSubArray(eq, 1, 19, y);

$$1,\ T_{a\,b} = \rho\,u_a\,u_b$$

$$2,\ P_{a\,b} = u\,u_{a\,b} + g_{a\,b}$$

$$3,\ P^a{}_b\,u^b = 0$$

$$4,\ dX^a = u^b\,X^a{}_{;b}$$

$$5,\ du^a = u^b\,u^a{}_{;b}$$

$$6,\ u_{a\,;b} = \frac{1}{3}\,\theta\,P_{a\,b} + \sigma_{a\,b} + \omega_{a\,b} - du_a\,u_b$$

$$7,\ \theta = u^a{}_{;a}$$

$$8,\ \sigma_{a\,b} = \frac{1}{2}\,P_a{}^c\,P_b{}^d\,u_{c\,;d} + \frac{1}{2}\,P_b{}^c\,P_a{}^d\,u_{c\,;d} - \frac{1}{3}\,\theta\,P_{a\,b}$$

$$9,\ \omega_{a\,b} = \frac{1}{2}\,P_a{}^c\,P_b{}^d\,u_{c\,;d} - \frac{1}{2}\,P_b{}^c\,P_a{}^d\,u_{c\,;d}$$

$$10,\ \omega^{a} = \frac{1}{2}\,\eta^{a\,b\,c\,d}\,u_{b}\,\omega_{c\,d}$$

$$11,\ \omega_{a\,b} = \eta_{a\,b\,e\,f}\,\omega^{e}\,u^{f}$$

$$12,\ \omega^{2} = \frac{1}{2}\,\omega^{a\,b}\,\omega_{a\,b}$$

13, "iff(iff(omega[-a,-b] = 0,omega[-a]),omega = 0)"

$$14,\ \omega_{a}{}^{c}\,\omega_{c}{}^{b} = -\omega^{2}\,P_{a}{}^{b} + \omega^{b}\,\omega_{a}$$

$$15,\ \frac{1}{2}\,u_{b\,;a} - \frac{1}{2}\,u_{a\,;b} = \frac{1}{2}\,du_{a}\,u_{b} - \frac{1}{2}\,du_{b}\,u_{a} + \omega^{a\,b}$$

$$16,\ -\frac{1}{6}\,u_{c}\,u_{a\,;b} + \frac{1}{6}\,u_{c}\,u_{b\,;a} + \frac{1}{6}\,u_{b}\,u_{a\,;c} - \frac{1}{6}\,u_{b}\,u_{c\,;a} - \frac{1}{6}\,u_{a}\,u_{b\,;c} + \frac{1}{6}\,u_{a}\,u_{c\,;b} = 0$$

$$17,\ \sigma_{a\,b} = 0$$

$$18,\ u_{b\,;a} = -u_{a}\,u_{b\,;c}\,u^{c} + \frac{1}{3}\,\theta\,h_{a\,b} + \omega_{a\,b}$$

$$19,\ u^{a}{}_{;c\,;d} - u^{a}{}_{;d\,;c} = R^{a}{}_{b\,c\,d}\,u^{b} \qquad \textbf{(1.25)}$$

>

>