```
> restart;
> with(Riemann):with(Canon):
> with(TensorPack) : CDF(0) : CDS(index) :
```

Chapter XX

Tensor analysis using indices - Senovilla et al. - Shearfree for dust - page 2 - if σ_{ab}=0 =>ωΘ=0

Author: Peter Huf

file 2a-eqs 13-14

```
> read "EFE" : read "SFE" :read "fids" :read "Seneqs1c" :
```

****

Equations 13 - zero vorticity

****

```
>
> eq[13] := convert('omega[-a,-b] = 0 iff omega[-a] iff omega = 0 ', string);
```

$$eq_{13} := \text{"iff(iff(omega[-a,-b] = 0,omega[-a]),omega = 0)"} \quad \textbf{(1.1)}$$

****

Equations 14ab - properties of vorticity

****

From eq 11 the identities are derived

```
> eq[14] := omega[-a, c]·omega[-c, b] = omega[-a]·omega[b] - omega·omega·P[-a, b] :
    T(eq[14 ]);
```

$$\omega_a{}^c \omega_c{}^b = -\omega^2 P_a{}^b + \omega_a \omega^b \quad \textbf{(1.2)}$$

```
> eq[14 a] := omega[-a, c]·omega[-c, b] = omega[-a]·omega[b] - omega·omega·P[-a, b] :
    T(eq[14 a]);
```

$$\omega_a{}^c \omega_c{}^b = -\omega^2 P_a{}^b + \omega_a \omega^b \quad \textbf{(1.3)}$$

```
> eq[14 b] := omega[-a, c]·omega[-c, d]·omega[-d, b] =- omega·omega·omega[-a, b] :
    T(eq[14 b]);
```

$$\omega_a{}^c \omega_c{}^d \omega_d{}^b = -\omega^2 \omega_a{}^b \quad \textbf{(1.4)}$$

Proof of eq 14a (proved):

```
> proof[14 a] := eq[14 a] : T(%);
```

$$\omega_a{}^c \omega_c{}^b = -\omega^2 P_a{}^b + \omega_a \omega^b \quad \textbf{(1.5)}$$

```
> proof[14 al] := lhs(proof[14 a]) : T(%);
```

(1.6)

$$\omega_{a}{}^{c}\omega_{c}{}^{b} \quad (1.6)$$

> *subs*(-*b* = *c*, *eq*[11]) : *T*(%);

$$\omega_{a}{}^{c} = \eta_{a}{}^{c}{}_{ef}\omega^{e}u^{f} \quad (1.7)$$

> *proof*[14 *al*] := *TEDS*(*subs*(-*b* = *c*, *eq*[11]), *proof*[14 *al*]) : *T*(%);

$$\omega_{c}{}^{b}\eta_{a}{}^{c}{}_{ef}\omega^{e}u^{f} \quad (1.8)$$

> *subs*(-*b* = *b*, -*a* =-*c*, *e* = *h*, *f* = *i*, *eq*[11]) : *T*(%);

$$\omega_{c}{}^{b} = \eta_{c}{}^{b}{}_{hi}\omega^{h}u^{i} \quad (1.9)$$

> *proof*[14 *al*] := *TEDS*(*subs*(-*b* = *b*, -*a* =-*c*, *e* = *h*, *f* = *i*, *eq*[11]), *proof*[14 *al*]) : *T*(%);

$$\eta_{a}{}^{c}{}_{ef}\omega^{e}u^{f}\eta_{c}{}^{b}{}_{hi}\omega^{h}u^{i} \quad (1.10)$$

an identity for eta is:

> eta[-*a*, *c*, -*e*, -*f*]·eta[-*c*, *b*, -*h*, -*i*] = *g*[-*a*, -*d*]·*g*[-*e*, -*j*]·*g*[-*f*, -*k*]·*g*[*b*, *l*]·eta[*d*, *c*, *j*, *k*]·eta[-*c*, -*l*, -*h*, -*i*] : *T*(%);

$$\eta_{a}{}^{c}{}_{ef}\eta_{c}{}^{b}{}_{hi} = g_{ad}g_{ej}g_{fk}g^{bl}\eta^{dcjk}\eta_{clhi} \quad (1.11)$$

swapping the first 2 indices (equivalent to multiplication by -1)

> *temp* := eta[-*a*, *c*, -*e*, -*f*]·eta[-*c*, *b*, -*h*, -*i*] =-*g*[-*a*, -*d*]·*g*[-*e*, -*j*]·*g*[-*f*, -*k*]·*g*[*b*, *l*]·eta[*c*, *d*, *j*, *k*]·eta[-*c*, -*l*, -*h*, -*i*] : *T*(%);

$$\eta_{a}{}^{c}{}_{ef}\eta_{c}{}^{b}{}_{hi} = -g_{ad}g_{ej}g_{fk}g^{bl}\eta^{cdjk}\eta_{clhi} \quad (1.12)$$

>

>

now substitute in the main equation:

> *proof*[14 *al*] := *TEDS*(temp, *proof*[14 *al*]) : *T*(%);

$$-\omega^{e}u^{f}\omega^{h}u^{i}g_{ad}g_{ej}g_{fk}g^{bl}\eta^{cdjk}\eta_{clhi} \quad (1.13)$$

> *temp* := eta[*c*, *d*, *j*, *k*]·eta[-*c*, -*l*, -*h*, -*i*] =-6·*antisymm*(delta[*d*, -*l*]·delta[*j*, -*h*]·delta[*k*, -*i*], *d*, *k*) : *T*(%);

$$\eta^{cdjk}\eta_{clhi} = -\delta^{k}{}_{l}\delta^{d}{}_{h}\delta^{j}{}_{i} + \delta^{j}{}_{l}\delta^{d}{}_{h}\delta^{k}{}_{i} + \delta^{k}{}_{l}\delta^{j}{}_{h}\delta^{d}{}_{i} - \delta^{j}{}_{l}\delta^{k}{}_{h}\delta^{d}{}_{i} - \delta^{d}{}_{l}\delta^{j}{}_{h}\delta^{k}{}_{i} + \delta^{d}{}_{l}\delta^{k}{}_{h}\delta^{j}{}_{i} \quad (1.14)$$

> *proof*[14 *al*] := *TEDS*(temp, *proof*[14 *al*]) : *T*(%);

$$\omega^{e}u^{f}\omega^{h}u^{i}g_{ad}g_{ej}g_{fk}g^{bl}\left(\delta^{k}{}_{l}\delta^{d}{}_{h}\delta^{j}{}_{i} - \delta^{j}{}_{l}\delta^{d}{}_{h}\delta^{k}{}_{i} - \delta^{k}{}_{l}\delta^{j}{}_{h}\delta^{d}{}_{i} + \delta^{j}{}_{l}\delta^{k}{}_{h}\delta^{d}{}_{i} + \delta^{d}{}_{l}\delta^{j}{}_{h}\delta^{k}{}_{i} - \delta^{d}{}_{l}\delta^{k}{}_{h}\delta^{j}{}_{i}\right) \quad (1.15)$$

> *expand*(*proof*[14 *al*]) : *T*(%);

$$\delta^{d}{}_{h}\delta^{j}{}_{i}\delta^{k}{}_{l}g^{bl}g_{ad}g_{ej}g_{fk}\omega^{e}\omega^{h}u^{f}u^{i} - \delta^{d}{}_{h}\delta^{j}{}_{l}\delta^{k}{}_{i}g^{bl}g_{ad}g_{ej}g_{fk}\omega^{e}\omega^{h}u^{f}u^{i} \quad (1.16)$$

$$-\delta^{d}{}_{i}\delta^{j}{}_{h}\delta^{k}{}_{l}g^{b\,l}g_{a\,d}g_{e\,j}g_{f\,k}\omega^{e}\omega^{h}u^{f}u^{i}$$
$$+\delta^{d}{}_{i}\delta^{j}{}_{l}\delta^{k}{}_{h}g^{b\,l}g_{a\,d}g_{e\,j}g_{f\,k}\omega^{e}\omega^{h}u^{f}u^{i}$$
$$+\delta^{d}{}_{l}\delta^{j}{}_{h}\delta^{k}{}_{i}g^{b\,l}g_{a\,d}g_{e\,j}g_{f\,k}\omega^{e}\omega^{h}u^{f}u^{i}$$
$$-\delta^{d}{}_{l}\delta^{j}{}_{i}\delta^{k}{}_{h}g^{b\,l}g_{a\,d}g_{e\,j}g_{f\,k}\omega^{e}\omega^{h}u^{f}u^{i}$$

> *temp* := *Absorbd*(*Absorbd*(*Absorbd*(*proof*[14 *al*]))) : *T*(%);

$$g^{b\,l}g_{a\,h}g_{e\,i}g_{f\,l}\omega^{e}\omega^{h}u^{f}u^{i}-g^{b\,l}g_{a\,h}g_{e\,l}g_{f\,i}\omega^{e}\omega^{h}u^{f}u^{i}-g^{b\,l}g_{a\,i}g_{e\,h}g_{f\,l}\omega^{e}\omega^{h}u^{f}u^{i}+g^{b\,l}g_{a\,i}g_{e\,l}g_{f\,h}\omega^{e}\omega^{h}u^{f}u^{i}+g^{b\,l}g_{a\,l}g_{e\,h}g_{f\,i}\omega^{e}\omega^{h}u^{f}u^{i}-g^{b\,l}g_{a\,l}g_{e\,i}g_{f\,h}\omega^{e}\omega^{h}u^{f}u^{i} \quad \mathbf{(1.17)}$$

> *temp* := *Absorbg*(*Absorbg*(*Absorbg*(*Absorbg*(*Absorbg*(*temp*))))) : *T*(%);

$$g_{a}{}^{b}g_{e\,h}g_{f\,i}\omega^{e}\omega^{h}u^{f}u^{i}-g_{a}{}^{b}g_{e\,i}g_{f\,h}\omega^{e}\omega^{h}u^{f}u^{i}+\omega^{b}\omega^{h}u_{h}u_{a}-\omega^{b}\omega_{a}u_{i}u^{i}-\omega_{h}\omega^{h}u^{b}u_{a}+\omega_{i}\omega_{a}u^{b}u^{i} \quad \mathbf{(1.18)}$$

> *subs*(*a* = *i*, *id*[1]) : *T*(%);

$$u^{i}u_{i}=-1 \quad \mathbf{(1.19)}$$

> *temp* := *expand*(*TEDS*(subs(a = i, id[1]), temp,)) : *T*(%);

$$g_{a}{}^{b}g_{e\,h}g_{f\,i}\omega^{e}\omega^{h}u^{f}u^{i}-g_{a}{}^{b}g_{e\,i}g_{f\,h}\omega^{e}\omega^{h}u^{f}u^{i}+\omega^{b}\omega^{h}u_{h}u_{a}-\omega_{h}\omega^{h}u^{b}u_{a}+\omega_{i}\omega_{a}u^{b}u^{i}+\omega^{b}\omega_{a} \quad \mathbf{(1.20)}$$

> *temp* := *expand*(*TEDS*(subs(omega[−*i*]·*u*[*i*] = 0), temp,)) : *T*(%);

$$g_{a}{}^{b}g_{e\,h}g_{f\,i}\omega^{e}\omega^{h}u^{f}u^{i}-g_{a}{}^{b}g_{e\,i}g_{f\,h}\omega^{e}\omega^{h}u^{f}u^{i}+\omega^{b}\omega^{h}u_{h}u_{a}-\omega_{h}\omega^{h}u^{b}u_{a}+\omega^{b}\omega_{a} \quad \mathbf{(1.21)}$$

> *temp* := *expand*(*TEDS*(subs(omega[*h*]·*u*[−*h*] = 0), temp,)) : *T*(%);

$$g_{a}{}^{b}g_{e\,h}g_{f\,i}\omega^{e}\omega^{h}u^{f}u^{i}-g_{a}{}^{b}g_{e\,i}g_{f\,h}\omega^{e}\omega^{h}u^{f}u^{i}-\omega_{h}\omega^{h}u^{b}u_{a}+\omega^{b}\omega_{a} \quad \mathbf{(1.22)}$$

> *proof*[14 *al*] := *temp* : *T*(%);

$$g_{a}{}^{b}g_{e\,h}g_{f\,i}\omega^{e}\omega^{h}u^{f}u^{i}-g_{a}{}^{b}g_{e\,i}g_{f\,h}\omega^{e}\omega^{h}u^{f}u^{i}-\omega_{h}\omega^{h}u^{b}u_{a}+\omega^{b}\omega_{a} \quad \mathbf{(1.23)}$$

> *show*(*id*);

$$table\Big(\Big[1=\left(u_{a}u_{-a}=-1\right),2=\left(g_{-a,\,a}=4\right),3=\left(g_{a,\,-a}=4\right),4=\left(du_{a}=u_{b}u_{a,\,b''}\right),5=\big(\Theta=u_{a,\,-a''}\big),6=\Big(\sigma_{-a,\,-b}=\frac{1}{2}P_{-a,\,c}P_{-b,\,d}u_{-c,\,-d''}-\frac{1}{6}\Theta P_{-a,\,-b}+\frac{1}{2}P_{-b,\,c}P_{-a,\,d}u_{-c,\,-d''}-\frac{1}{6}\Theta P_{-b,\,-a}\Big),7=\Big(\omega_{-a,\,-b}=\frac{1}{2}P_{-a,\,c}P_{-b,\,d}u_{c,\,d''}-\frac{1}{2}P_{-b,\,c}P_{-a,\,d}u_{c,\,d''}\Big),9=\Big(\omega_{a} \quad \mathbf{(1.24)}$$

$= \frac{1}{2}\eta_{a,b,c,d}u_{-b}\omega_{-c,-d}\Big), 8 = \Big(du_{-a}u_{-b} + u_{-a,-b}" = \frac{1}{3}\Theta P_{-a,-b} + \sigma_{-a,-b}$

$+ \omega_{-a,-b}\Big), 11 = (P_{a,-b}u_b = 0), 10 = (P_{a,-b}P_{b,-c} = P_{a,-c}), 13 = \Big(g_{-a,-b} = \frac{1}{2}g_{-a,-b}$

$+ \frac{1}{2}g_{-b,-a}\Big), 12 = \Big(P_{-a,-b} = \frac{1}{2}g_{-a,-b} + u_{-a}u_{-b} + \frac{1}{2}g_{-b,-a}\Big)\Big]\Big)$

> deq[1] := P[-a, b]=g[-a, b] + u[-a]·u[b] : T(deq[1]);

$$P_a{}^b = u_a u^b + g_a{}^b \quad (1.25)$$

> proof[14 ar] := rhs(eq[14 a]) : T(%);

$$-\omega^2 P_a{}^b + \omega^b \omega_a \quad (1.26)$$

> temp := expand(TEDS(deq[1], proof[14 ar])) : T(%);

$$-\omega^2 u^b u_a - \omega^2 g_a{}^b + \omega^b \omega_a \quad (1.27)$$

> proof[14 ar] := expand(TEDS(omega·omega = omega[-h]·omega[h], temp)) : T(%);

$$-\omega_h \omega^h u^b u_a - g_a{}^b \omega_h \omega^h + \omega^b \omega_a \quad (1.28)$$

> proof[14 al] : T(%);

$$g_a{}^b g_{eh} g_{fi} \omega^e \omega^h u^f u^i - g_a{}^b g_{ei} g_{fh} \omega^e \omega^h u^f u^i - \omega_h \omega^h u^b u_a + \omega^b \omega_a \quad (1.29)$$

> proof[14 al] − proof[14 ar];

$$g_{-a,b} g_{-e,-h} g_{-f,-i} \omega_e \omega_h u_f u_i - g_{-a,b} g_{-e,-i} g_{-f,-h} \omega_e \omega_h u_f u_i + g_{-a,b} \omega_h \omega_{-h} \quad (1.30)$$

> subs(g[b,-a] = g[-a, b], %);

$$g_{-a,b} g_{-e,-h} g_{-f,-i} \omega_e \omega_h u_f u_i - g_{-a,b} g_{-e,-i} g_{-f,-h} \omega_e \omega_h u_f u_i + g_{-a,b} \omega_h \omega_{-h} \quad (1.31)$$

thus proving eq14a***
****

> eq[14 a] := omega[-a, c]·omega[-c, b] = omega[-a]·omega[b] - omega·omega·P[-a, b] :
    T(eq[14 a]);

$$\omega_a{}^c \omega_c{}^b = -\omega^2 P_a{}^b + \omega^b \omega_a \quad (1.32)$$

> eq[14 b] := omega[-a, c]·omega[-c, d]·omega[-d, b] =-omega·omega·omega[-a, b] :
    T(eq[14 b]);

$$\omega_a{}^c \omega_c{}^d \omega_d{}^b = -\omega^2 \omega_a{}^b \quad (1.33)$$

Proof of eq 14b (proved):

> lhs(eq[14 b]) : T(%);

$$\omega_a{}^c \omega_c{}^d \omega_d{}^b \quad (1.34)$$

subs b=d in eq14a gives

> temp := subs(b = d, eq[14 a]) : T(%);

(1.35)

$$\omega_{a}{}^{c}\omega_{c}{}^{d}=-\omega^{2}P_{a}{}^{d}+\omega^{d}\omega_{a} \quad \textbf{(1.35)}$$

> *proof*[14 *bl*] := *expand*(*TEDS*(*temp*, *lhs*(*eq*[14 *b*]))) : *T*(%);

$$-\omega^{2}P_{a}{}^{d}\omega_{d}{}^{b}+\omega^{d}\omega_{a}\omega_{d}{}^{b} \quad \textbf{(1.36)}$$

> omega[*h*]·*u*[−*h*]=0;

$$\omega_{h}u_{-h}=0 \quad \textbf{(1.37)}$$

> *proof*[14 *bl*] := *expand*(*TEDS*(omega[−*d*, *b*]·omega[*d*]=0, *proof*[14 *bl*])) : *T*(%);

$$-\omega^{2}P_{a}{}^{d}\omega_{d}{}^{b} \quad \textbf{(1.38)}$$

> *proof*[14 *bl*] := *expand*(*TEDS*(P[−a, d]=g[−*a*, *d*]+*u*[−*a*]·*u*[*d*], *proof*[14 *bl*])) : *T*(%);

$$-\omega^{2}\omega_{d}{}^{b}u^{d}u_{a}-\omega^{2}g_{a}{}^{d}\omega_{d}{}^{b} \quad \textbf{(1.39)}$$

> *proof*[14 *bl*] := *expand*(*TEDS*(omega[−*d*, *b*]·*u*[*d*]=0, *proof*[14 *bl*])) : *T*(%);

$$-\omega^{2}g_{a}{}^{d}\omega_{d}{}^{b} \quad \textbf{(1.40)}$$

> *Absorbg*(*proof*[14 *bl*]) : *T*(%);

$$-\omega^{2}\omega_{a}{}^{b} \quad \textbf{(1.41)}$$

thus proving eq14b**

>

> **save** *eq*, "Seneqs2a" :

go to page 2b

> **read** "Seneqs2a";

$$eq := table\Big(\Big[1=\big(TensorPack\text{:-}T_{-a,\,-b}=\rho\,u_{-a}u_{-b}\big),\,2=\big(P_{-a,\,-b}=u_{-a}u_{-b}+g_{-a,\,-b}\big),\,3=\big(P_{a,\,-b}u_{b}=0\big),\,4=\big(dX_{a}=u_{b}X_{a,\,-B}\big),\,5=\big(du_{a}=u_{b}u_{a,\,-B}\big),\,6=\Big(u_{-a,\,-B}=\frac{1}{3}\,\theta\,P_{-a,\,-b}+\sigma_{-a,\,-b}+\omega_{-a,\,-b}-du_{-a}u_{-b}\Big),\,7=\big(\theta=u_{a,\,-A}\big),\,10\,b=\Big(\eta_{-f,\,-g,\,-a,\,-e}\omega_{a}u_{e}=\frac{1}{2}\,\eta_{-f,\,-g,\,-a,\,-e}\eta_{a,\,b,\,c,\,d}u_{-b}\omega_{-c,\,-d}u_{e}\Big),\,9=\Big(\omega_{-a,\,-b}=\frac{1}{2}\,P_{-a,\,c}P_{-b,\,d}u_{-c,\,-D}-\frac{1}{2}\,P_{-b,\,c}P_{-a,\,d}u_{-c,\,-D}\Big),\,8=\Big(\sigma_{-a,\,-b}=\frac{1}{2}\,P_{-a,\,c}P_{-b,\,d}u_{-c,\,-D}+\frac{1}{2}\,P_{-b,\,c}P_{-a,\,d}u_{-c,\,-D}-\frac{1}{3}\,\theta\,P_{-a,\,-b}\Big),\,11=\big(\omega_{-a,\,-b}=\eta_{-a,\,-b,\,-e,\,-f}\omega_{e}u_{f}\big),\,10=\Big(\omega_{a}=\frac{1}{2}\,\eta_{a,\,b,\,c,\,d}u_{-b}\omega_{-c,\,-d}\Big),\,13=\text{"iff(iff(omega[-a,-b] = 0,omega[-a]),omega = 0)"},\,12 \quad \textbf{(1.42)}$$

$= \left(\omega^2 = \frac{1}{2}\,\omega_{a,\,b}\,\omega_{-a,\,-b}\right),\ 14 = \left(\omega_{-a,\,c}\,\omega_{-c,\,b} = -\omega^2\,P_{-a,\,b} + \omega_b\,\omega_{-a}\right),\ 11\ m1 = \left(\omega_{a,\,-b}\right.$

$\left. = \eta_{a,\,-b,\,-c,\,-d}\,u_d\,\omega_c\right),\ 14\ a = \left(\omega_{-a,\,c}\,\omega_{-c,\,b} = -\omega^2\,P_{-a,\,b} + \omega_b\,\omega_{-a}\right),\ 14\ b$

$= \left(\omega_{-a,\,c}\,\omega_{-c,\,d}\,\omega_{-d,\,b} = -\omega^2\,\omega_{-a,\,b}\right),\ 10\ a = \left(\omega_b = \frac{1}{2}\,\eta_{b,\,e,\,f,\,g}\,u_{-e}\,\omega_{-f,\,-g}\right),\ 11\ m2$

$= \left(\omega_{a,\,b} = \eta_{a,\,b,\,-e,\,-f}\,u_f\,\omega_e\right),\ 11\ m = \left(\omega_{-a,\,-b} = \eta_{-a,\,-b,\,-c,\,-d}\,u_d\,\omega_c\right),\ 12\ b = \left(\omega^2\right.$

$\left. = \omega_a\,\omega_{-a}\right),\ 7\ a = \left(\theta = u_{d,\,-\mathrm{D}}\right),\ 12\ a = \left(\omega^2 = \frac{1}{2}\,\omega_{a,\,b}\,\omega_{-a,\,-b}\right)\Big]\Big)$

> PrintSubArray(eq, 1, 14, y);

$$1,\ T_{a\,b} = \rho\,u_a\,u_b$$

$$2,\ P_{a\,b} = u\,u_{a\,b} + g_{a\,b}$$

$$3,\ P^{a}{}_{b}\,u^{b} = 0$$

$$4,\ dX^{a} = u^{b}\,X^{a}{}_{;b}$$

$$5,\ du^{a} = u^{b}\,u^{a}{}_{;b}$$

$$6,\ u_{a\,;b} = \frac{1}{3}\,\theta\,P_{a\,b} + \sigma_{a\,b} + \omega_{a\,b} - du_a\,u_b$$

$$7,\ \theta = u^{a}{}_{;a}$$

$$8,\ \sigma_{a\,b} = \frac{1}{2}\,P_a{}^{c}\,P_b{}^{d}\,u_{c\,;d} + \frac{1}{2}\,P_b{}^{c}\,P_a{}^{d}\,u_{c\,;d} - \frac{1}{3}\,\theta\,P_{a\,b}$$

$$9,\ \omega_{a\,b} = \frac{1}{2}\,P_a{}^{c}\,P_b{}^{d}\,u_{c\,;d} - \frac{1}{2}\,P_b{}^{c}\,P_a{}^{d}\,u_{c\,;d}$$

$$10,\ \omega^{a} = \frac{1}{2}\,\eta^{a\,b\,c\,d}\,u_b\,\omega_{c\,d}$$

$$11,\ \omega_{a\,b} = \eta_{a\,b\,e\,f}\,\omega^{e}\,u^{f}$$

$$12,\ \omega^2 = \frac{1}{2}\,\omega^{a\,b}\,\omega_{a\,b}$$

13, "iff(iff(omega[-a,-b] = 0,omega[-a]),omega = 0)"

$$14,\ \omega_a{}^{c}\,\omega_c{}^{b} = -\omega^2\,P_a{}^{b} + \omega^{b}\,\omega_a \qquad \textbf{(1.43)}$$

>

>