```
> restart;
> with(Riemann):with(Canon):
> with(TensorPack) : CDF(0) : CDS(index) :
```

Chapter XX
Tensor analysis using indices - Senovilla et al. - Shearfree for dust
if σ_{ab}=0 =>$\omega\Theta$=0
Author: Peter Huf
file 1c-equations 11-12

```
> read "EFE" : read "SFE" :read "fids" :read "Seneqs1b" : read "deqs1b" :read deltaid :
```

****

****

Equation 11 Relation between voticity vector and tensor

****

```
>
```

It is possible to show that this equation can be inverted to achieve
$\omega_{ab} = \eta_{abef}\omega^{e}u^{f}$

We do so as follows. Firstly multiply equation 10 by the velocity u^{e}:

```
> eq[10 a] := lhs(eq[10])·u[e] = rhs(eq[10])·u[e] : T(%);
```

$$\omega^{a} u^{e} = \frac{1}{2}\eta^{abcd} u_{b}\omega_{cd} u^{e} \qquad \textbf{(1.1)}$$

Then multiply by η_{fgae}:

```
> eq[10 b] := eta[-f,-g,-a,-e]·lhs(eq[10 a]) = eta[-f,-g,-a,-e]·rhs(eq[10 a]) : T(%);
```

$$\eta_{fgae}\omega^{a} u^{e} = \frac{1}{2}\eta_{fgae}\eta^{abcd} u_{b}\omega_{cd} u^{e} \qquad \textbf{(1.2)}$$

At this point we need to look relook at some critical identities of η^{abcd} and $\delta^{a}{}_{b}$
(It might be better to refer to another file for the following identities:)

```
>
> deltaid[1 a] := delta[a,-b]·u[b] = u[a] : T(%);
```

(1.3)

$$\delta^{a}{}_{b} u^{b} = u^{a} \tag{1.3}$$

```
> deltaid[1 b] := delta[a,-b]·u[-a]=u[-b] : T(%);
```

$$\delta^{a}{}_{b} u_{a} = u_{b} \tag{1.4}$$

```
> deltaid[1 c] := delta[a,-b]·omega[-a,-c]=omega[-b,-c] : T(%);
```

$$\delta^{a}{}_{b} \omega_{a c} = \omega_{b c} \tag{1.5}$$

```
> deltaid[1 d] := delta[a,-b]·omega[-c,-a]=omega[-c,-b] : T(%);
```

$$\delta^{a}{}_{b} \omega_{c a} = \omega_{c b} \tag{1.6}$$

```
> etaid[1] := eta[-f,-g,-a,-e]=eta[-a,-f,-g,-e] : T(%);
```

$$\eta_{f g a e} = \eta_{a f g e} \tag{1.7}$$

```
> temp1 := TEDS(etaid[1], eq[10 b]) : T(%);
```

$$\omega^{a} u^{e} \eta_{a f g e} = \frac{1}{2} \eta^{a b c d} u_{b} \omega_{c d} u^{e} \eta_{a f g e} \tag{1.8}$$

```
> temp2 := eta[-a,-f,-g,-e]·eta[a, b, c, d]=-6·antisymm(delta[b,-f]·delta[c,-g]·delta[d,
    -e], b, d) : T(%);
```

$$\eta_{a f g e} \eta^{a b c d} = -\delta^{c}{}_{f} \delta^{d}{}_{g} \delta^{b}{}_{e} + \delta^{d}{}_{f} \delta^{c}{}_{g} \delta^{b}{}_{e} + \delta^{b}{}_{f} \delta^{d}{}_{g} \delta^{c}{}_{e} - \delta^{b}{}_{f} \delta^{c}{}_{g} \delta^{d}{}_{e} - \delta^{d}{}_{f} \delta^{b}{}_{g} \delta^{c}{}_{e} + \delta^{c}{}_{f} \delta^{b}{}_{g} \delta^{d}{}_{e} \tag{1.9}$$

```
> temp3 := expand(TEDS(temp2, temp1)) : T(%);
```

$$\omega^{a} u^{e} \eta_{a f g e} = -\frac{1}{2} \delta^{b}{}_{e} \delta^{c}{}_{f} \delta^{d}{}_{g} \omega_{c d} u^{e} u_{b} + \frac{1}{2} \delta^{b}{}_{e} \delta^{c}{}_{g} \delta^{d}{}_{f} \omega_{c d} u^{e} u_{b} + \frac{1}{2} \delta^{b}{}_{f} \delta^{c}{}_{e} \delta^{d}{}_{g} \omega_{c d} u^{e} u_{b} - \frac{1}{2} \delta^{b}{}_{f} \delta^{c}{}_{g} \delta^{d}{}_{e} \omega_{c d} u^{e} u_{b} - \frac{1}{2} \delta^{b}{}_{g} \delta^{c}{}_{e} \delta^{d}{}_{f} \omega_{c d} u^{e} u_{b} + \frac{1}{2} \delta^{b}{}_{g} \delta^{c}{}_{f} \delta^{d}{}_{e} \omega_{c d} u^{e} u_{b} \tag{1.10}$$

```
> temp4 := Absorbd(Absorbd(Absorbd(temp3))) : T(%);
```

$$\omega^{a} u^{e} \eta_{a f g e} = -\frac{1}{2} \omega_{f g} u^{e} u_{e} + \frac{1}{2} \omega_{g f} u^{e} u_{e} + \frac{1}{2} \omega_{e g} u^{e} u_{f} - \frac{1}{2} \omega_{g e} u^{e} u_{f} - \frac{1}{2} \omega_{e f} u^{e} u_{g} + \frac{1}{2} \omega_{f e} u^{e} u_{g} \tag{1.11}$$

```
> temp5 := expand(TEDS(u[e]·omega[-e,-f]=0, temp4)) : T(%);
```

$$\omega^{a} u^{e} \eta_{a f g e} = -\frac{1}{2} \omega_{f g} u^{e} u_{e} + \frac{1}{2} \omega_{g f} u^{e} u_{e} + \frac{1}{2} \omega_{e g} u^{e} u_{f} - \frac{1}{2} \omega_{g e} u^{e} u_{f} + \frac{1}{2} \omega_{f e} u^{e} u_{g} \tag{1.12}$$

```
> temp6 := expand(TEDS(u[e]·omega[-g,-e]=0, temp5)) : T(%);
```

(1.13)

$$\omega^{a} u^{e} \eta_{afge} = -\frac{1}{2}\, \omega_{fg} u^{e} u_{e} + \frac{1}{2}\, \omega_{gf} u^{e} u_{e} + \frac{1}{2}\, \omega_{eg} u^{e} u_{f} + \frac{1}{2}\, \omega_{fe} u^{e} u_{g} \qquad (1.13)$$

```
> temp7 := expand(TEDS(u[e]·omega[-e,-g]=0, temp6)) : T(%);
```

$$\omega^{a} u^{e} \eta_{afge} = -\frac{1}{2}\, \omega_{fg} u^{e} u_{e} + \frac{1}{2}\, \omega_{gf} u^{e} u_{e} + \frac{1}{2}\, \omega_{fe} u^{e} u_{g} \qquad (1.14)$$

```
> temp8 := expand(TEDS(u[e]·omega[-f,-e]=0, temp7)) : T(%);
```

$$\omega^{a} u^{e} \eta_{afge} = -\frac{1}{2}\, \omega_{fg} u^{e} u_{e} + \frac{1}{2}\, \omega_{gf} u^{e} u_{e} \qquad (1.15)$$

```
> temp9 := expand(TEDS(u[e]·u[-e]=-1, temp8)) : T(%);
```

$$\omega^{a} u^{e} \eta_{afge} = \frac{1}{2}\, \omega_{fg} - \frac{1}{2}\, \omega_{gf} \qquad (1.16)$$

```
> temp10 := expand(TEDS(omega[-g,-f]=-omega[-f,-g], temp9)) : T(%);
```

$$\omega^{a} u^{e} \eta_{afge} = \omega_{fg} \qquad (1.17)$$

```
> temp11 := expand(TEDS(eta[-a,-f,-g,-e]=eta[-f,-g,-a,-e], temp10)) : T(%);
```

$$\eta_{fgae}\, \omega^{a} u^{e} = \omega_{fg} \qquad (1.18)$$

```
> temp12 := subs(a=-a, e=-e, f=-f, g=-g, temp11) : T(%);
```

$$\eta^{fgae}\, \omega_{a} u_{e} = \omega^{fg} \qquad (1.19)$$

completing the proof of eq11

and so we have

```
> eq[11] := rhs(temp12) = lhs(temp12) : T(%);
```

$$\omega^{fg} = \eta^{fgae}\, \omega_{a} u_{e} \qquad (1.20)$$

****

****

Equation 11b Vorticity vector and tensor are orthogonal

****

```
>
```

We proceed to show $\omega_{ab}\,\omega^{b} = 0$

```
> expr := omega[-a,-b]·omega[b] : T(%);
```

$$\omega_{ab}\, \omega^{b} \qquad (1.21)$$

```
> eq[10] := omega[a] = (1/2)·eta[a, b, c, d]·u[-b]·omega[-c,-d] : T(%);
```

$$\omega^{a} = \frac{1}{2}\, \eta^{abcd}\, u_{b}\, \omega_{cd} \qquad (1.22)$$

eq[11m] written out explicitly is:

> eq[11 m] := omega[-a,-b] = eta[-a,-b,-c,-d]·u[d]·omega[c] : T(%);

$$\omega_{a\,b} = \eta_{a\,b\,c\,d}\, u^{\,d}\, \omega^{\,c} \tag{1.23}$$

> eq[10 a] := subs(b = e, c = f, d = g, a = b, eq[10]) : T(%);

$$\omega^{\,b} = \frac{1}{2}\, \eta^{\,b\ e\ f\ g}\, u_{e}\, \omega_{f g} \tag{1.24}$$

> temp := expand(TEDS(eq[11 m], expr)) : T(%);

$$\omega^{\,b}\, \eta_{a\,b\,c\,d}\, u^{\,d}\, \omega^{\,c} \tag{1.25}$$

> temp2 := expand(TEDS(eq[10 a], temp)) : T(%);

$$\frac{1}{2}\, \eta_{a\,b\,c\,d}\, u^{\,d}\, \omega^{\,c}\, \eta^{\,b\ e\ f\ g}\, u_{e}\, \omega_{f g} \tag{1.26}$$

so we have

> temp2 := expr = temp2 : T(%);

$$\omega_{a\,b}\, \omega^{\,b} = \frac{1}{2}\, \eta_{a\,b\,c\,d}\, u^{\,d}\, \omega^{\,c}\, \eta^{\,b\ e\ f\ g}\, u_{e}\, \omega_{f g} \tag{1.27}$$

> temp3 := expand(TEDS(eta[-a,-b,-c,-d] = -eta[-b,-a,-c,-d], temp2)) : T(%);

$$\omega_{a\,b}\, \omega^{\,b} = -\frac{1}{2}\, u^{\,d}\, \omega^{\,c}\, \eta^{\,b\ e\ f\ g}\, u_{e}\, \omega_{f g}\, \eta_{b\,a\,c\,d} \tag{1.28}$$

> temp4 := expand(TEDS(eta[b, e, f, g]·eta[-b,-a,-c,-d] = -6·antisymm(delta[e,-a]·delta[f,-c]·delta[g,-d], e, g), temp3,)) : T(%);

$$\begin{aligned}\omega_{a\,b}\, \omega^{\,b} = {} & \frac{1}{2}\, \delta^{\,e}{}_{a}\, \delta^{\,f}{}_{c}\, \delta^{\,g}{}_{d}\, \omega^{\,c}\, \omega_{f g}\, u^{\,d}\, u_{e} - \frac{1}{2}\, \delta^{\,e}{}_{a}\, \delta^{\,f}{}_{d}\, \delta^{\,g}{}_{c}\, \omega^{\,c}\, \omega_{f g}\, u^{\,d}\, u_{e} \\ & - \frac{1}{2}\, \delta^{\,e}{}_{c}\, \delta^{\,f}{}_{a}\, \delta^{\,g}{}_{d}\, \omega^{\,c}\, \omega_{f g}\, u^{\,d}\, u_{e} + \frac{1}{2}\, \delta^{\,e}{}_{c}\, \delta^{\,f}{}_{d}\, \delta^{\,g}{}_{a}\, \omega^{\,c}\, \omega_{f g}\, u^{\,d}\, u_{e} \\ & + \frac{1}{2}\, \delta^{\,e}{}_{d}\, \delta^{\,f}{}_{a}\, \delta^{\,g}{}_{c}\, \omega^{\,c}\, \omega_{f g}\, u^{\,d}\, u_{e} - \frac{1}{2}\, \delta^{\,e}{}_{d}\, \delta^{\,f}{}_{c}\, \delta^{\,g}{}_{a}\, \omega^{\,c}\, \omega_{f g}\, u^{\,d}\, u_{e}\end{aligned} \tag{1.29}$$

> temp5 := Absorbd(Absorbd(Absorbd(temp4))) : T(%);

$$\begin{aligned}\omega_{a\,b}\, \omega^{\,b} = {} & \frac{1}{2}\, \omega^{\,c}\, \omega_{c d}\, u^{\,d}\, u_{a} - \frac{1}{2}\, \omega^{\,c}\, \omega_{d c}\, u^{\,d}\, u_{a} - \frac{1}{2}\, \omega^{\,c}\, \omega_{a d}\, u^{\,d}\, u_{c} \\ & + \frac{1}{2}\, \omega^{\,c}\, \omega_{d a}\, u^{\,d}\, u_{c} + \frac{1}{2}\, \omega^{\,c}\, \omega_{a c}\, u^{\,d}\, u_{d} - \frac{1}{2}\, \omega^{\,c}\, \omega_{c a}\, u^{\,d}\, u_{d}\end{aligned} \tag{1.30}$$

> temp6 := expand(TEDS(u[d]·u[-d] = -1, temp5)) : T(%);

$$\begin{aligned}\omega_{a\,b}\, \omega^{\,b} = {} & \frac{1}{2}\, \omega^{\,c}\, \omega_{c d}\, u^{\,d}\, u_{a} - \frac{1}{2}\, \omega^{\,c}\, \omega_{d c}\, u^{\,d}\, u_{a} - \frac{1}{2}\, \omega^{\,c}\, \omega_{a d}\, u^{\,d}\, u_{c} \\ & + \frac{1}{2}\, \omega^{\,c}\, \omega_{d a}\, u^{\,d}\, u_{c} - \frac{1}{2}\, \omega^{\,c}\, \omega_{a c} + \frac{1}{2}\, \omega^{\,c}\, \omega_{c a}\end{aligned} \tag{1.31}$$

> temp7 := expand(TEDS(u[d]·omega[-a,-d] = 0, temp6)) : T(%);

$$\omega_{a b}\,\omega^{b}=\frac{1}{2}\,\omega^{c}\,\omega_{c d}\,u^{d}\,u_{a}-\frac{1}{2}\,\omega^{c}\,\omega_{d c}\,u^{d}\,u_{a}+\frac{1}{2}\,\omega^{c}\,\omega_{d a}\,u^{d}\,u_{c}-\frac{1}{2}\,\omega^{c}\,\omega_{a c}+\frac{1}{2}\,\omega^{c}\,\omega_{c a} \quad (1.32)$$

> temp8 := expand(TEDS(u[d]·omega[-d,-c]=0, temp7)) : T(%);

$$\omega_{a b}\,\omega^{b}=\frac{1}{2}\,\omega^{c}\,\omega_{c d}\,u^{d}\,u_{a}+\frac{1}{2}\,\omega^{c}\,\omega_{d a}\,u^{d}\,u_{c}-\frac{1}{2}\,\omega^{c}\,\omega_{a c}+\frac{1}{2}\,\omega^{c}\,\omega_{c a} \quad (1.33)$$

> temp9 := expand(TEDS(u[d]·omega[-d,-a]=0, temp8,)) : T(%);

$$\omega_{a b}\,\omega^{b}=\frac{1}{2}\,\omega^{c}\,\omega_{c d}\,u^{d}\,u_{a}-\frac{1}{2}\,\omega^{c}\,\omega_{a c}+\frac{1}{2}\,\omega^{c}\,\omega_{c a} \quad (1.34)$$

> temp10 := expand(TEDS(u[d]·omega[-c,-d]=0, temp9,)) : T(%);

$$\omega_{a b}\,\omega^{b}=-\frac{1}{2}\,\omega^{c}\,\omega_{a c}+\frac{1}{2}\,\omega^{c}\,\omega_{c a} \quad (1.35)$$

> temp11 := expand(TEDS(omega[-c,-a]=-omega[-a,-c], temp10)) : T(%);

$$\omega_{a b}\,\omega^{b}=-\omega^{c}\,\omega_{a c} \quad (1.36)$$

> temp12 := subs(c=b, temp11) : T(%);

$$\omega_{a b}\,\omega^{b}=-\omega_{a b}\,\omega^{b} \quad (1.37)$$

> temp13 := temp12 + omega[-a,-b]·omega[b] : T(%);

$$2\,\omega_{a b}\,\omega^{b}=0 \quad (1.38)$$

proving the equation

>

> ##\
##

****

Equation 12 Definition of the vorticity scalar

****

We define the vorticity / rotation scalar:

> eq[12] := $\omega^2 = \left(\frac{1}{2}\right)$ · omega[a, b]·omega[-a,-b] : T(%);

$$\omega^{2}=\frac{1}{2}\,\omega^{a b}\,\omega_{a b} \quad (1.39)$$

> eq[11 m] := omega[-a,-b]=eta[-a,-b,-c,-d]·u[d]·omega[c] : T(%);

$$\omega_{a b}=\eta_{a b c d}\,u^{d}\,\omega^{c} \quad (1.40)$$

> temp := expand(TEDS(eq[11 m], eq[12])) : T(%);

$$\omega^{2}=\frac{1}{2}\,\omega^{a b}\,\eta_{a b c d}\,u^{d}\,\omega^{c} \quad (1.41)$$

> eq[11 m1] := raise(omega[-a,-b]=eta[-a,-b,-c,-d]·u[d]·omega[c], a) : T(%);

(1.42)

$$\omega^{a}{}_{b} = \eta^{a}{}_{bcd} u^{d} \omega^{c} \quad \textbf{(1.42)}$$

```
> eq[11 m2] := subs(c = e, d = f, raise(eq[11 m1], b)) : T(%);
```

$$\omega^{ab} = \eta^{ab}{}_{ef} u^{f} \omega^{e} \quad \textbf{(1.43)}$$

```
> temp := expand(TEDS(eq[11 m2], temp)) : T(%);
```

$$\omega^2 = \frac{1}{2} \eta_{abcd} u^{d} \omega^{c} \eta^{ab}{}_{ef} u^{f} \omega^{e} \quad \textbf{(1.44)}$$

```
> temp := raise(temp, e) : temp := raise(temp, f) : T(%);
```

$$\omega^2 = \frac{1}{2} \eta_{abcd} u^{d} \omega^{c} \eta^{abef} u_{f} \omega_{e} \quad \textbf{(1.45)}$$

Now we use the identity:

$$\eta_{abcd} \eta^{abef} = -4 \delta_{[c}{}^{e} \delta_{d]}{}^{f}$$

written explicitly:

####################################

```
> e1 := eta[-a,-b,-c,-d]·eta[a, b, e, f] = -4· antisymm(delta[-c, e]·delta[-d, f],-c,-d) :
    T(%);
```

$$\eta_{abcd} \eta^{abef} = -2 \delta_{c}{}^{e} \delta_{d}{}^{f} + 2 \delta_{d}{}^{e} \delta_{c}{}^{f} \quad \textbf{(1.46)}$$

```
> temp;
```

$$\omega^2 = \frac{1}{2} \eta_{-a,\,-b,\,-c,\,-d} u_{d} \omega_{c} \eta_{a,\,b,\,e,\,f} u_{-f} \omega_{-e} \quad \textbf{(1.47)}$$

```
> temp := expand(TEDS(e1, temp))
```

$$temp := \omega^2 = -\delta_{-c,\,e} \delta_{-d,\,f} \omega_{c} \omega_{-e} u_{d} u_{-f} + \delta_{-c,\,f} \delta_{-d,\,e} \omega_{c} \omega_{-e} u_{d} u_{-f} \quad \textbf{(1.48)}$$

```
> T(%);
```

$$\omega^2 = -\delta_{c}{}^{e} \delta_{d}{}^{f} \omega^{c} \omega_{e} u^{d} u_{f} + \delta_{c}{}^{f} \delta_{d}{}^{e} \omega^{c} \omega_{e} u^{d} u_{f} \quad \textbf{(1.49)}$$

```
> temp := Absorbd(Absorbd(rhs(temp)));
```

$$temp := -\omega_{e} \omega_{-e} u_{f} u_{-f} + \omega_{f} \omega_{-e} u_{e} u_{-f} \quad \textbf{(1.50)}$$

```
> T(%);
```

$$-\omega^{e} \omega_{e} u^{f} u_{f} + \omega^{f} \omega_{e} u^{e} u_{f} \quad \textbf{(1.51)}$$

```
> e2 := subs(a = f, id[1]) : T(%);
```

$$u^{f} u_{f} = -1 \quad \textbf{(1.52)}$$

```
> temp := expand(TEDS(e2, temp)) : T(%);
```

$$\omega^{f} \omega_{e} u^{e} u_{f} + \omega^{e} \omega_{e} \quad \textbf{(1.53)}$$

```
> e2 := expand(TEDS(u[e]·omega[-e]=0, temp)) : T(%);
```

$$\omega^{e}\,\omega_{e} \tag{1.54}$$

hence we have shown that (Senovilla eqn 12)

$\omega^2 = \frac{1}{2}\,\omega^{a\,b}\,\omega_{a\,b} = \omega^{e}\,\omega_{e}$

****
****

```
>
> ######################################not used
    below#############################################
> eq[11] := omega[-a,-b]=eta[-a,-b,-e,-f]·omega[e]·u[f] : T(%);
```

$$\omega_{a\,b} = \eta_{a\,b\,e\,f}\,\omega^{e}\,u^{f} \tag{1.55}$$

```
> raise(eq[10], a) : T(%);
```

$$\omega_{a} = \frac{1}{2}\,\eta_{a}{}^{b\;c\;d}\,u_{b}\,\omega_{c\,d} \tag{1.56}$$

```
> eq[12 a] := eq[12];
```

$$eq_{12\,a} := \omega^2 = \frac{1}{2}\,\omega_{a,\,b}\,\omega_{-a,\,-b} \tag{1.57}$$

```
> eq[12 b] := ω^2 = omega[a]·omega[-a] : T(%);
```

$$\omega^2 = \omega^{a}\,\omega_{a} \tag{1.58}$$

****

End of page 1c - to equation 12

****

```
>
> save eq, "Seneqs1c";
> read "Seneqs1c" :
> show(eq) :
>
> PrintSubArray(eq, 1, 12, y);
```

$$1,\ T_{a\,b} = \rho\,u_{a}\,u_{b}$$

$$2,\ P_{a\,b} = u\,u_{a\,b} + g_{a\,b}$$

$$3,\ P^{a}{}_{b}\,u^{b} = 0$$

$$4,\ dX^{a} = u^{b}\,X^{a}{}_{;b}$$

$$5,\ du^{a} = u^{b}\,u^{a}{}_{;b}$$

$$6, u_{a;b} = \frac{1}{3}\,\theta P_{ab} + \sigma_{ab} + \omega_{ab} - du_a u_b$$

$$7, \theta = u^{a}{}_{;a}$$

$$8, \sigma_{ab} = \frac{1}{2} P_a{}^c P_b{}^d u_{c;d} + \frac{1}{2} P_b{}^c P_a{}^d u_{c;d} - \frac{1}{3}\,\theta P_{ab}$$

$$9, \omega_{ab} = \frac{1}{2} P_a{}^c P_b{}^d u_{c;d} - \frac{1}{2} P_b{}^c P_a{}^d u_{c;d}$$

$$10, \omega^{a} = \frac{1}{2}\,\eta^{abcd} u_b \omega_{cd}$$

$$11, \omega_{ab} = \eta_{abef}\,\omega^{e} u^{f}$$

$$12, \omega^2 = \frac{1}{2}\,\omega^{ab}\omega_{ab} \qquad \textbf{(1.59)}$$

>

>