```
> restart;
> with(Riemann):with(Canon):
> with(TensorPack) : CDF(0) : CDS(index) :
```

Chapter XX
Tensor analysis using indices - Senovilla et al. - Shearfree for dust
if $\sigma_{a\,b}$=0 =>ωΘ=0
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file 1

Introduction

The theorem $\sigma_{a\,b}$=0 =>ωΘ=0 has been shown using tetrads. There is no complete proof using a 1+3 covariant formalism.

In this file we follow the equations outlined by

Senovilla, J.M.M., Sopuerta, C.F., Szekeres, P. Theorems on shear-free perfect fluids with their Newtonian analogues, Gen.Rel.Grav, 30, 389-411 (1998)

with the assumptions for **dust**
i.e p=0, du=0, shear=0, viscosity=0

1. General results

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Equation 1 (SSSeq1)

****

The energy stress tensor for a perfect fluidreads:

```
> ET_PF := T[-a,-b] = rho·u[-a]·u[-b] + p·P[-a,-b] : T(ET_PF);
```

$$T_{a\,b} = \rho\, u\, u_{a\,b} + P\, p_{a\,b} \tag{2.1}$$

where for dust, p=0, and so

```
> eq[1] := subs(p=0, ET_PF) : T(eq[1]);
```

$$T_{a\,b} = \rho\, u_a\, u_b \tag{2.2}$$

where
u_a is the unit velocity vector field
ρ = energy density
p = pressure

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Equation 2 Projection tensor and identities (SSSeq2)

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The projection tensor, P, is defined as:

```
> deq[1] := P[-a,-b] = g[-a,-b] + u[-a]·u[-b] : T(deq[1]);
```

$$P_{a\,b} = u\,u_{a\,b} + g_{a\,b} \qquad (2.3)$$

```
> eq[2] := deq[1] :
```

$P_{a\,b}$ defines the projection tensor, the projector orthogonal to **u**.
i.e. $P_{a\,b}$ creates the component of **u** in the direction of b.

>

There are several properties that can be proved from the definition at this point.

We use known identities:

```
> id[1] := u[a]·u[-a] = -1 : T(id[1]);
```

$$u^{a}\,u_{a} = -1 \qquad (2.4)$$

```
> id[2] := g[a,-a] = 4 : T(id[2]);
```

$$g^{a}{}_{a} = 4 \qquad (2.5)$$

2a. Symmetry of the projection tensor:

```
> deq[1] : T(deq[1]);
```

$$P_{a\,b} = u\,u_{a\,b} + g_{a\,b} \qquad (2.6)$$

Due to the symmetry of g, we can see immediately that

```
> deq[2b] := P[-a,-b] = P[-b,-a] : T(%);
```

$$P_{a\,b} = P_{b\,a} \qquad (2.7)$$

****

Equations 3abc Other properties of the projection tensor (SSSeqs3abc)

****

We prove several simple identities related **to** P :

Firstly we aim to prove SSSeq3b

```
> P[a,-a] = 3 : T(%);
```

$$P^{a}{}_{a} = 3 \qquad (2.8)$$

Commencing with the original identity:

```
> T(deq[1]);
```

$$P_{a\,b} = u\,u_{a\,b} + g_{a\,b} \qquad (2.9)$$

We raise the index a:

```
> deq[1 b] := raise(deq[1], a) : T(%);
```

$$P^{a}{}_{b} = u^{a} u_{b} + g^{a}{}_{b} \quad (2.10)$$

and then contract a on b:

```
> deq[1 c] := contract(deq[1b], b, a) : T(%);
```

$$P^{b}{}_{b} = u^{b} u_{b} + g^{b}{}_{b} \quad (2.11)$$

We substitute the velocity and metric identies:

```
> deq[1 d] := TELS(id[1], deq[1 c]) : T(deq[1 d]);
```

$$P^{b}{}_{b} = -1 + g^{b}{}_{b} \quad (2.12)$$

```
> deq[1 e] := TELS(id[2], deq[1 d]) : T(%);
```

$$P^{b}{}_{b} = 3 \quad (2.13)$$

```
>
```

We move to SSSeq3a

```
> SSSeq3a := P[a,-b]·P[b,-c]=P[a,-c] : T(%);
```

$$P^{a}{}_{b} P^{b}{}_{c} = P^{a}{}_{c} \quad (2.14)$$

We start with the LHS:

```
> deq[3 a] := P[a,-b]·P[b,-c] : T(%);
```

$$P^{a}{}_{b} P^{b}{}_{c} \quad (2.15)$$

Substituting identities:

```
> deq[3 b] := TEDS(deq[1 b], deq[3 a]) : T(%);
```

$$P^{b}{}_{c} \left(u^{a} u_{b} + g^{a}{}_{b}\right) \quad (2.16)$$

```
> deq[3 d] := TELS(deq[1 b], deq[3 b]) : T(%);
```

$$\left(u^{a} u_{b} + g^{a}{}_{b}\right) \left(u^{b} u_{c} + g^{b}{}_{c}\right) \quad (2.17)$$

```
> deq[3 e] := expand(deq[3 d]) : T(%);
```

$$u^{a} u^{b} u_{b} u_{c} + g^{a}{}_{b} u^{b} u_{c} + g^{b}{}_{c} u^{a} u_{b} + g^{a}{}_{b} g^{b}{}_{c} \quad (2.18)$$

```
> deq[3 f] := Absorbg(deq[3e]) : T(%);
```

$$u^{a} u^{b} u_{b} u_{c} + 2\, u^{a} u_{c} + g^{a}{}_{c} \quad (2.19)$$

```
> id[1 b] := u[-b]·u[b] =-1;
```

$$id_{b} := u_{b} u_{-b} = -1 \quad (2.20)$$

```
> deq[3 g] := TEDS(id[1 b], deq[3 f]) : T(%);
```

$$u^{a} u_{c} + g^{a}{}_{c} \quad (2.21)$$

leads to

by definition, this is $P^{a}{}_{c}$.

So we add to the array of equations

```
> deq[3 h] := P[a,-c] = P[a,-b]·P[b,-c] : T(%);
```

$$P^{a}{}_{c} = P^{a}{}_{b} P^{b}{}_{c} \quad (2.22)$$

Now we try to show equation 3c: $u^{b} P^{a}{}_{b} = 0$

```
> deq[3 c] := deq[1 b]·u[b] : T(%);
```

$$u^{b} P^{a}{}_{b} = u^{b} \left(u^{a} u_{b} + g^{a}{}_{b}\right) \quad (2.23)$$

```
> deq[3 c] := expand(deq[3 c]) : T(%);
```

$$u^{b} P^{a}{}_{b} = u^{a} u^{b} u_{b} + g^{a}{}_{b} u^{b} \quad (2.24)$$

```
> deq[3 c] := Absorbg(TEDS(subs(a = b, id[1]), deq[3 c])) : eq[3] := deq[3 c] :
    T(deq[3 c]);
```

$$u^{b} P^{a}{}_{b} = 0 \quad (2.25)$$

which proves the equation 3c.

```
>
```

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Equation 4- Definition of a time derivative of a tensor

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For any vector X, the time derivative

```
> eq[4] := dX[a] = u[b]·X[a,-B] : T(%);
```

$$dX^{a} = u^{b} X^{a}{}_{;b} \quad (2.26)$$

This can apply to any tensor. S
See application to acceleration in eq5

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Equation 5 Definition of acceleration

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We move now to various definitions, firstly for acceleration, du:

```
> eq[5] := du[a] = u[b]·u[a,-B] : T(%);
```

$$du^{a} = u^{b} u^{a}{}_{;b} \quad (2.27)$$

and now for various kinematic quatities and relationships:

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Equation 6 Decomposition of covariant derivative of velocity

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```
>
```

```
> eq[6] := u[-a,-B] = (1/3)·theta·P[-a,-b] + sigma[-a,-b] + omega[-a,-b]
     - du[-a]·u[-b] : T(%);
```

$$u_{a\,;b} = \frac{1}{3}\,\theta\,P_{a\,b} + \sigma_{a\,b} + \omega_{a\,b} - du_{a}\,u_{b} \tag{2.28}$$

The proof is important and widely used, but is common in the literature and will not be formally proved at this point (for a detailed proof see see Ellis (1970)).

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End of page 1 - to equation 6

****

```
> save deq, "deqs1a";
> save eq, "Seneqs1a";
> read "Seneqs1a" :
> PrintSubArray(eq, 1, 6, y);
```

$$1,\ T_{a\,b} = \rho\,u_{a}\,u_{b}$$

$$2,\ P_{a\,b} = u\,u_{a\,b} + g_{a\,b}$$

$$3,\ u^{b}\,P^{a}{}_{b} = 0$$

$$4,\ dX^{a} = u^{b}\,X^{a}{}_{;b}$$

$$5,\ du^{a} = u^{b}\,u^{a}{}_{;b}$$

$$6,\ u_{a\,;b} = \frac{1}{3}\,\theta\,P_{a\,b} + \sigma_{a\,b} + \omega_{a\,b} - du_{a}\,u_{b} \tag{2.29}$$

```
>
>
>
```