```
> restart;with(Riemann):with(TensorPack): with(Canon):CDF(0): CDS(index):
```

Chapter XX

Tensor analysis using indices - Senovilla et al. - Shearfree for

acceleration parallel to vorticity

if $\sigma_{a\,b}$=0 =>$\omega\Theta$=0

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file 2d:eq60-revisit1

In this file we continue to follow the equations outlined by Senovilla et al. (2007) with the assumptions for acceleration parallel to vorticity
i.e

```
> read "EFE" : read "SFE" :read "fids" :read "Seneqs80" :
```

proof of eq60:

```
> eq[60] := P[a,-b]·omega[b, d,-D] = eta[-a,-b,-c,-d]·u[b]·ifdu[d, C] : T(%);
```

$$P^{a}{}_{b}\,\omega^{b\ d}{}_{;d} = \eta_{a\,b\,c\,d}\,u^{b}\,ifdu^{d\ ;c} \tag{1.1}$$

where

```
> defifdu := ifdu = du/Psi : T(%);
```

$$ifdu = \frac{du}{\Psi} \tag{1.2}$$

```
> defifdua := ifdu[a] = du[a]/Psi : T(%);
```

$$ifdu^{a} = \frac{du^{a}}{\Psi} \tag{1.3}$$

and the following assumption is operational, where Ψ is an arbitrary non-vanishing function:

```
> assumption := du = Psi·omega : T(%);
```

$$du = \Psi\,\omega \tag{1.4}$$

```
> assumption2 := omega[a] = du[a]/Psi : T(%);
```

$$\omega^{a} = \frac{du^{a}}{\Psi} \tag{1.5}$$

proof:

****
****

```
> temp := subs(b =-d, a =-b, eq[11]) : T(%);
```

$$\omega^{b d} = \eta^{b d}{}_{e f} \omega^{e} u^{f} \tag{1.6}$$

```
> temp2 := cod(temp,-d) : T(%);
```

$$\omega^{b d}{}_{;d} = \eta^{b d}{}_{e f} \omega^{e} u^{f}{}_{;d} + \eta^{b d}{}_{e f} \omega^{e}{}_{;d} u^{f} + \eta^{b d}{}_{e f ;d} \omega^{e} u^{f} \tag{1.7}$$

```
> temp3 := TEDS(eta[b, d,-e,-f,-D] = 0, temp2) : T(%);
```

$$\omega^{b d}{}_{;d} = \eta^{b d}{}_{e f} \omega^{e} u^{f}{}_{;d} + \eta^{b d}{}_{e f} \omega^{e}{}_{;d} u^{f} \tag{1.8}$$

```
> temp4 := expand(P[-a,-b]·temp3) : T(%);
```

$$P_{a b} \omega^{b d}{}_{;d} = P_{a b} \eta^{b d}{}_{e f} \omega^{e} u^{f}{}_{;d} + P_{a b} \eta^{b d}{}_{e f} \omega^{e}{}_{;d} u^{f} \tag{1.9}$$

```
> temp5 := lhs(temp4) = Absorbg(TEDS(P[-a,-b] = g[-a,-b] + u[-a]·u[-b],
    rhs(temp4))) : T(%);
```

$$P_{a b} \omega^{b d}{}_{;d} = \eta^{b d}{}_{e f} \omega^{e} u_{a} u_{b} u^{f}{}_{;d} + \eta^{b d}{}_{e f} \omega^{e}{}_{;d} u^{f} u_{a} u_{b} + \eta_{a}{}^{d}{}_{e f} \omega^{e} u^{f}{}_{;d} + \eta_{a}{}^{d}{}_{e f} \omega^{e}{}_{;d} u^{f} \tag{1.10}$$

```
> temp6 := TEDS(eta[b, d,-e,-f]·u[-b]·u[f] = 0, temp5) : T(%);
```

$$P_{a b} \omega^{b d}{}_{;d} = \eta^{b d}{}_{e f} \omega^{e} u_{a} u_{b} u^{f}{}_{;d} + \eta_{a}{}^{d}{}_{e f} \omega^{e} u^{f}{}_{;d} + \eta_{a}{}^{d}{}_{e f} \omega^{e}{}_{;d} u^{f} \tag{1.11}$$

```
> eq[60] : T(%);
```

$$P^{a}{}_{b} \omega^{b d}{}_{;d} = \eta_{a b c d} u^{b} ifdu^{d ;c} \tag{1.12}$$

```
>
```

```
> temp7 := subs(a = d, cod(defifdua, c)) : T(%);
```

$$ifdu^{d ;c} = \frac{du^{d ;c}}{\Psi} - \frac{du^{d} \Psi^{;c}}{\Psi^{2}} \tag{1.13}$$

subs for omega in temp6:

```
> temp8 := TEDS(subs(a = e, assumption2), temp6) : T(%);
```

$$P_{a b} \omega^{b d}{}_{;d} = \frac{\eta^{b d}{}_{e f} u_{a} u_{b} u^{f}{}_{;d} du^{e} + \eta_{a}{}^{d}{}_{e f} \omega^{e}{}_{;d} u^{f} \Psi + \eta_{a}{}^{d}{}_{e f} u^{f}{}_{;d} du^{e}}{\Psi} \tag{1.14}$$

```
> subs(d = e, C =-D, temp7) : T(%);
```

$$ifdu^{e}{}_{;d} = \frac{du^{e}{}_{;d}}{\Psi} - \frac{du^{e} \Psi_{;d}}{\Psi^{2}} \tag{1.15}$$

```
> temp9 := cod(subs(a = e, assumption2),-d) : T(%);
```

(1.16)

$$\omega^{e}{}_{;d} = \frac{du^{e}{}_{;d}}{\Psi} - \frac{du^{e}\Psi_{;d}}{\Psi^2} \tag{1.16}$$

```
> temp10 := TEDS(temp9, temp8) : T(%);
```

$$P_{ab}\omega^{bd}{}_{;d} = \frac{1}{\Psi^2}\left(\eta^{bd}{}_{ef}u_a u_b u^{f}{}_{;d} du^{e}\Psi + \eta_a{}^{d}{}_{ef}u^{f}{}_{;d} du^{e}\Psi + \Psi du^{e}{}_{;d}\eta_a{}^{d}{}_{ef}u^{f} - \Psi_{;d} du^{e}\eta_a{}^{d}{}_{ef}u^{f}\right) \tag{1.17}$$

****

The last term in temp10 can be shown to be the first term on the rhs of eq60b.
Looking at the terms on the RHS
****

The first term:

```
> temp101 := op(1, op(2, op(2, temp10))) : T(%);
```

$$\eta^{bd}{}_{ef}u_a u_b u^{f}{}_{;d} du^{e}\Psi \tag{1.18}$$

```
> temp102 := subs(sigma[-a,-b]=0, a=-f, b=d, B=D, eq[6]) : T(%);
```

$$u^{f}{}_{;d} = \frac{1}{3}\theta P^{f}{}_{d} + \omega^{f}{}_{d} - du^{f}u_d \tag{1.19}$$

```
> temp103 := expand(TEDS(temp102, temp101)) : T(%);
```

$$\frac{1}{3}\Psi\theta P^{f}{}_{d} du^{e}\eta^{bd}{}_{ef}u_a u_b - \Psi du^{e} du^{f}\eta^{bd}{}_{ef}u_a u_b u_d + \Psi du^{e}\eta^{bd}{}_{ef}\omega^{f}{}_{d}u_a u_b \tag{1.20}$$

Eliminating product of symmetric and antisymmetric terms:

```
> temp104 := expand(TEDS(P[f,-d]·eta[b, d,-e,-f]=0, temp103)) : T(%);
```

$$-\Psi du^{e} du^{f}\eta^{bd}{}_{ef}u_a u_b u_d + \Psi du^{e}\eta^{bd}{}_{ef}\omega^{f}{}_{d}u_a u_b \tag{1.21}$$

```
> temp105 := expand(TEDS(u[-b]·u[-d]·eta[b, d,-e,-f]=0, temp104)) : T(%);
```

$$\Psi du^{e}\eta^{bd}{}_{ef}\omega^{f}{}_{d}u_a u_b \tag{1.22}$$

```
> temp106 := TEDS(eta[b, d,-e,-f]·u[-b]·omega[f,-d]=omega[-e], temp105) : T(%);
```

$$\Psi du^{e}u_a\omega_e \tag{1.23}$$

(check sign of above step - should be ok)

```
> temp107 := TEDS(du[e]·omega[-e]=Psi·ω^2, temp106) : T(%);
```

$$\Psi^2 u_a\omega^2 \tag{1.24}$$

```
> term1 := temp107 : T(%);
```

$$\Psi^2 u_a\omega^2 \tag{1.25}$$

This must be timelike, and so result in zero component in a spacelike projection.

end of first term**

****

The second term:

> $temp201 := op(2, op(2, op(2, temp10))) : T(\%);$

$$\eta_{a}{}^{d}{}_{ef} u^{f}{}_{;d} du^{e} \Psi \quad \textbf{(1.26)}$$

> $temp202 := subs(\mathrm{sigma}[-a,-b]=0, a=-f, b=d, B=\mathrm{D}, eq[6]) : T(\%);$

$$u^{f}{}_{;d} = \frac{1}{3}\theta P^{f}{}_{d} + \omega^{f}{}_{d} - du^{f} u_{d} \quad \textbf{(1.27)}$$

> $temp203 := expand(TEDS(temp202, temp201)) : T(\%);$

$$\frac{1}{3}\Psi\theta P^{f}{}_{d} du^{e} \eta_{a}{}^{d}{}_{ef} - \Psi du^{e} du^{f} \eta_{a}{}^{d}{}_{ef} u_{d} + \Psi du^{e} \eta_{a}{}^{d}{}_{ef} \omega^{f}{}_{d} \quad \textbf{(1.28)}$$

Eliminating product of symmetric and antisymmetric terms:

> $temp204 := expand(TEDS(P[f,-d]\cdot\mathrm{eta}[-a, d,-e,-f]=0, temp203)) : T(\%);$

$$-\Psi du^{e} du^{f} \eta_{a}{}^{d}{}_{ef} u_{d} + \Psi du^{e} \eta_{a}{}^{d}{}_{ef} \omega^{f}{}_{d} \quad \textbf{(1.29)}$$

> $temp205 := expand(TEDS(du[e]\cdot du[f]\cdot\mathrm{eta}[-a, d,-e,-f]=0, temp204)) : T(\%);$

$$\Psi du^{e} \eta_{a}{}^{d}{}_{ef} \omega^{f}{}_{d} \quad \textbf{(1.30)}$$

> $temp106 := TEDS(\mathrm{du}[\mathrm{e}]=\mathrm{Psi}\cdot\mathrm{omega}[e], temp205) : T(\%);$

$$\Psi^2 \eta_{a}{}^{d}{}_{ef} \omega^{f}{}_{d} \omega^{e} \quad \textbf{(1.31)}$$

(check sign of above step ?)
Due tot eh eta tensor, the terms is orthogonal to both the vorticity vector and the plane of the vorticity tenso. Hence term 2 must be timelike (and so result in zero component in a spacelike projection) i.e. for some function K:

> $term2 := K\cdot u[-a] : T(\%);$

$$K u_{a} \quad \textbf{(1.32)}$$

end of second term***

****

The third term:

> $temp301 := op(3, op(2, op(2, temp10))) : T(\%);$

$$\Psi du^{e}{}_{;d} \eta_{a}{}^{d}{}_{ef} u^{f} \quad \textbf{(1.33)}$$

> $eq[54] : T(\%);$

$$-\frac{1}{2} P_{a}{}^{c} P_{b}{}^{d} du_{d;c} + \frac{1}{2} P_{b}{}^{c} P_{a}{}^{d} du_{d;c} = p'\theta\omega_{ab} \quad \textbf{(1.34)}$$

> $convert(eq[54], string);$

"-1/2*P[-a,c]*P[-b,d]*du[-d,-C]+1/2*P[-b,c]*P[-a,d]*du[-d,-C] = `p'`*theta*omega[-a,-b]" **(1.35)**

> $temp302 := subs(d=x, D=X, c=y, C=Y, b=e, a=d, eq[54]) : T(\%);$

$$-\frac{1}{2} P_{d}^{\ y} P_{e}^{\ x} du_{x;y} + \frac{1}{2} P_{e}^{\ y} P_{d}^{\ x} du_{x;y} = p' \theta\, \omega_{de} \tag{1.36}$$

> $temp303 := Absorbg(TEDS(P[-d, x] = g[-d, x] + u[-d] \cdot u[x], temp302)) : T(\%);$

$$-\frac{1}{2} P_{d}^{\ y} P_{e}^{\ x} du_{x;y} + \frac{1}{2} P_{e}^{\ y} du_{x;y} u^{x} u_{d} + \frac{1}{2} P_{e}^{\ y} du_{d;y} = p' \theta\, \omega_{de} \tag{1.37}$$

> $temp304 := Absorbg(TEDS(P[-e, y] = g[-e, y] + u[-e] \cdot u[y], temp303)) : T(\%);$

$$-\frac{1}{2} P_{d}^{\ y} P_{e}^{\ x} du_{x;y} + \frac{1}{2} du_{x;y} u^{x} u^{y} u_{d} u_{e} + \frac{1}{2} du_{x;e} u^{x} u_{d} + \frac{1}{2} du_{d;y} u^{y} u_{e} + \frac{1}{2} du_{d;e} = p' \theta\, \omega_{de} \tag{1.38}$$

> $temp305 := Absorbg(TEDS(P[-d, y] = g[-d, y] + u[-d] \cdot u[y], temp304)) : T(\%);$

$$-\frac{1}{2} P_{e}^{\ x} du_{x;y} u^{y} u_{d} - \frac{1}{2} P_{e}^{\ x} du_{x;d} + \frac{1}{2} du_{x;y} u^{x} u^{y} u_{d} u_{e} + \frac{1}{2} du_{x;e} u^{x} u_{d} + \frac{1}{2} du_{d;y} u^{y} u_{e} + \frac{1}{2} du_{d;e} = p' \theta\, \omega_{de} \tag{1.39}$$

> $temp306 := Absorbg(TEDS(P[-e, x] = g[-e, x] + u[-e] \cdot u[x], temp305)) : T(\%);$

$$-\frac{1}{2} du_{e;y} u^{y} u_{d} - \frac{1}{2} du_{x;d} u^{x} u_{e} - \frac{1}{2} du_{e;d} + \frac{1}{2} du_{x;e} u^{x} u_{d} + \frac{1}{2} du_{d;y} u^{y} u_{e} + \frac{1}{2} du_{d;e} = p' \theta\, \omega_{de} \tag{1.40}$$

> $temp307 := subs(e = -e, isolate(temp306, du[-e, -D])) : T(\%);$

$$du^{e}_{\ ;d} = -2\, p' \theta\, \omega_{d}^{\ e} - du^{e}_{\ ;y} u^{y} u_{d} + du_{d;y} u^{e} u^{y} - du_{x;d} u^{e} u^{x} + du_{x;e} u^{x} u_{d} + du_{d;e} \tag{1.41}$$

so to create the third term from this:

> $term31 := expand(\mathrm{eta}[-a, d, -e, -f] \cdot u[f] \cdot temp307 \cdot \mathrm{Psi}) : T(\%);$

$$\Psi\, du^{e}_{\ ;d} \eta_{a\ \ ef}^{\ d} u^{f} = -2\, \Psi\, p' \theta\, \eta_{a\ \ ef}^{\ d} \omega_{d}^{\ e} u^{f} - \Psi\, du^{e}_{\ ;y} \eta_{a\ \ ef}^{\ d} u^{f} u^{y} u_{d} + \Psi\, du_{d;y} \eta_{a\ \ ef}^{\ d} u^{e} u^{f} u^{y} - \Psi\, du_{x;d} \eta_{a\ \ ef}^{\ d} u^{e} u^{f} u^{x} + \Psi\, du_{x;e} \eta_{a\ \ ef}^{\ d} u^{f} u^{x} u_{d} + \Psi\, du_{d;e} \eta_{a\ \ ef}^{\ d} u^{f} \tag{1.42}$$

> $term32 := expand(TEDS(\mathrm{eta}[-a, d, -e, -f] \cdot u[f] \cdot u[-d] = 0, term31)) : T(\%);$

$$\Psi\, du^{e}_{\ ;d} \eta_{a\ \ ef}^{\ d} u^{f} = -2\, \Psi\, p' \theta\, \eta_{a\ \ ef}^{\ d} \omega_{d}^{\ e} u^{f} + \Psi\, du_{d;y} \eta_{a\ \ ef}^{\ d} u^{e} u^{f} u^{y} - \Psi\, du_{x;d} \eta_{a\ \ ef}^{\ d} u^{e} u^{f} u^{x} + \Psi\, du_{d;e} \eta_{a\ \ ef}^{\ d} u^{f} \tag{1.43}$$

> $term33 := expand(TEDS(\mathrm{eta}[-a, d, -e, -f] \cdot u[f] \cdot u[e] = 0, term32)) : T(\%);$

$$\Psi\, du^{e}_{\ ;d} \eta_{a\ \ ef}^{\ d} u^{f} = -2\, \Psi\, p' \theta\, \eta_{a\ \ ef}^{\ d} \omega_{d}^{\ e} u^{f} + \Psi\, du_{d;e} \eta_{a\ \ ef}^{\ d} u^{f} \tag{1.44}$$

> $term34 := expand(TEDS(\text{eta}[-a, d,-e,-f]\cdot \text{du}[-\text{d},-\text{E}] = -\text{eta}[-a, d,-e,-f]\cdot \text{du}[\text{e},-\text{D}], term33)) : T(\%);$

$$\Psi\, du^{e}{}_{;d}\, \eta_{a}{}^{d}{}_{ef}\, u^{f} = -2\,\Psi\, p'\theta\, \eta_{a}{}^{d}{}_{ef}\, \omega_{d}{}^{e}\, u^{f} - \Psi\, du^{e}{}_{;d}\, \eta_{a}{}^{d}{}_{ef}\, u^{f} \quad \textbf{(1.45)}$$

> $term35 := \frac{(term34 - op(2, op(2, term34)))}{2} : T(\%);$

$$\Psi\, du^{e}{}_{;d}\, \eta_{a}{}^{d}{}_{ef}\, u^{f} = -\Psi\, p'\theta\, \eta_{a}{}^{d}{}_{ef}\, \omega_{d}{}^{e}\, u^{f} \quad \textbf{(1.46)}$$

> $term36 := TEDS(\text{eta}[-a, d,-e,-f]\cdot u[f]\cdot \text{omega}[-d, e] = 2\cdot \text{omega}[-a], term35) : T(\%);$

$$\Psi\, du^{e}{}_{;d}\, \eta_{a}{}^{d}{}_{ef}\, u^{f} = -2\,\Psi\, p'\theta\, \omega_{a} \quad \textbf{(1.47)}$$

so we have

> $term3 := rhs(term36) : T(\%);$

$$-2\,\Psi\, p'\theta\, \omega_{a} \quad \textbf{(1.48)}$$

end of third term***

The fourth term:

> $term41 := op(4, op(2, op(2, temp10))) : T(\%);$

$$-\Psi_{;d}\, du^{e}\, \eta_{a}{}^{d}{}_{ef}\, u^{f} \quad \textbf{(1.49)}$$

> $term42 := TEDS(du[e] = \text{Psi}\cdot \text{omega}[e], term41) : T(\%);$

$$-\Psi_{;d}\, \eta_{a}{}^{d}{}_{ef}\, u^{f}\, \Psi\, \omega^{e} \quad \textbf{(1.50)}$$

Now from eq11:

> $term43 := subs(b = -d, eq[11]) : T(\%);$

$$\omega_{a}{}^{d} = \eta_{a}{}^{d}{}_{ef}\, \omega^{e}\, u^{f} \quad \textbf{(1.51)}$$

> $term44 := rhs(term43) = lhs(term43) : T(\%);$

$$\eta_{a}{}^{d}{}_{ef}\, \omega^{e}\, u^{f} = \omega_{a}{}^{d} \quad \textbf{(1.52)}$$

> $term45 := subs(\text{D} = -\text{D}, d = -d, TEDS(term44, term42)) : T(\%);$

$$-\Psi^{;d}\, \Psi\, \omega_{a\,d} \quad \textbf{(1.53)}$$

> $term4 := term45 : T(\%);$

$$-\Psi^{;d}\, \Psi\, \omega_{a\,d} \quad \textbf{(1.54)}$$

end of fourth term***

>

so combining the terms:

> $temp11 := lhs(temp10) = \frac{(term1 + term2 + term3 + term4)}{\Psi^2} : T(\%);$

(1.55)

$$P_{ab}\omega^{b\ d}{}_{;d}=\frac{\Psi^2 u_a \omega^2-2\Psi p'\theta\omega_a-\Psi^{;d}\Psi\omega_{ad}+K u_a}{\Psi^2} \qquad (1.55)$$

projecting with P[c,a]

> *temp12* := *expand*(*P*[*c*, *a*]·*temp11*) : *T*(%);

$$P^{c\ a}P_{ab}\omega^{b\ d}{}_{;d}=P^{c\ a}u_a\omega^2-\frac{2P^{c\ a}p'\theta\omega_a}{\Psi}-\frac{P^{c\ a}\Psi^{;d}\omega_{ad}}{\Psi}+\frac{P^{c\ a}K u_a}{\Psi^2} \qquad (1.56)$$

and substitute identities:

> *temp12a* := *subs*(*c* = *e*, *expand*(*TEDS*(*P*[*c*, *a*]·*u*[−*a*] = 0, *expand*(*TEDS*(*P*[*c*, *a*]·*P*[−*a*,−*b*] = *P*[*c*,−*b*], *temp12*))))) : *T*(%);

$$\omega^{b\ d}{}_{;d}P^{e}{}_{b}=-\frac{2P^{e\ a}p'\theta\omega_a}{\Psi}-\frac{P^{e\ a}\Psi^{;d}\omega_{ad}}{\Psi} \qquad (1.57)$$

> *temp12b* := *TEDS*(*P*[*e*, *a*]·omega[−*a*] = omega[*e*], *temp12a*) : *T*(%);

$$\omega^{b\ d}{}_{;d}P^{e}{}_{b}=-\frac{P^{e\ a}\Psi^{;d}\omega_{ad}}{\Psi}-\frac{2p'\theta\omega^{e}}{\Psi} \qquad (1.58)$$

> *temp12c* := *expand*(*subs*(*e* =−*a*, *TEDS*(*P*[*e*, *a*]·omega[−*a*,−*d*] = omega[*e*,−*d*], *temp12b*))) : *T*(%);

$$P_{ab}\omega^{b\ d}{}_{;d}=-\frac{2p'\theta\omega_a}{\Psi}-\frac{\Psi^{;d}\omega_{ad}}{\Psi} \qquad (1.59)$$

which is eq60 (second part)

> *eq*[60] := *convert*(*temp12c*, *string*) : *T*(%);

"P[-a,-b]*omega[b,d,-D] = -2*`p'`*theta*omega[-a]/Psi-1/Psi*Psi[D]*omega[-a,-d]" **(1.60)**

> *eq*[60] := *parse*("P[-a,-b]*omega[b,d,-D] = -2*`p'`*theta*omega[-a]/Psi-1/Psi*Psi[D]*omega[-a,-d]") : *T*(%);

$$P_{ab}\omega^{b\ d}{}_{;d}=-\frac{2p'\theta\omega_a}{\Psi}-\frac{\Psi^{;d}\omega_{ad}}{\Psi} \qquad (1.61)$$

>