```
> restart;with(Riemann):with(TensorPack): with(Canon):CDF(0): CDS(index):
```

Chapter XX
Tensor analysis using indices - Senovilla et al. - Shearfree for
acceleration parallel to vorticity

if $\sigma_{a\,b}$=0 =>$\omega\Theta$=0
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file 2d:eq55

In this file we continue to follow the equations outlined by Senovilla et al. (2007) with the assumptions for acceleration parallel to vorticity, in this case the proof of eq55

explanation of terms:

$PU = \mu + p$

time derivatives:
dotX=time derivative of entity X
noting eg.
$dotmu^{;b}$ = time derivative of $\mu^{;a}$
where as
mudot is the time derivative of μ
$\text{mudot}^{;a}$ = covariant derivative of mudot (mu=μ)

and so on...

```
> read "EFE" : read "SFE" :read "fids" :read "Seneqs31" :
```

We are attempting to prove eq55 of Senovilla et al. (2008) (same as eq 3.19 in Sopuerta (1998)):

$$> eq55SSS := P[a,-b]\cdot dotdu[b] = \left(`p'` - \frac{1}{3} - (\text{mu} + p)\cdot\left(\frac{`p''`}{`p'`}\right)\right)\cdot \text{theta}\cdot du[a] + \text{omega}[a,-b]\cdot du[b] + `p'`\cdot P[a,b]\cdot \text{theta}[-B] : T(\%);$$

$$P^{a}{}_{b}\, dotdu^{b} = \left(p' - \frac{1}{3} - \frac{(\mu+p)\,p''}{p'}\right)\theta\, du^{a} + \omega^{a}{}_{b}\, du^{b} + p'\, P^{a\,b}\,\theta_{;b} \qquad \textbf{(2.1.1)}$$

proof of eq55:
We commence with time dilation of equation 56:

> eq[56] := du[a] = `p'`·theta·u[a] − `p'`·mu[A] / (p + mu) : T(%);

$$du^{a} = p'\theta u^{a} - \frac{p'\mu^{;a}}{\mu + p} \tag{2.1.2}$$

****

> temp := factor((p + mu)·eq[56]) : T(%);

$$(\mu + p)\, du^{a} = p'\left(\theta u^{a}\mu + \theta u^{a} p - \mu^{;a}\right) \tag{2.1.3}$$

or rather

> temp2 := (p + mu)·du[a] = (p + mu)·`p'`·theta·u[a] − `p'`·mu[A] : T(%);

$$(\mu + p)\, du^{a} = p'\theta u^{a}(\mu + p) - p'\mu^{;a} \tag{2.1.4}$$

and we define

> defPU := mu + p = PU : T(%);

$$\mu + p = PU \tag{2.1.5}$$

> defPUr := PU = mu + p : T(%);

$$PU = \mu + p \tag{2.1.6}$$

> temp2b := TEDS(defPU, temp2) : T(%);

$$du^{a}\, PU = p'\theta u^{a}\, PU - p'\mu^{;a} \tag{2.1.7}$$

> temp3 := subs(a = b, A = B, dotT(temp2b)) : T(%);

$$PU\, dotdu^{b} + dotPU\, du^{b} = PU\, dotp'\theta u^{b} + PU\, p'\theta\, dotu^{b} + PU\, p'\, dottheta\, u^{b} + dotPU\, p'\theta u^{b} - dotp'\mu^{;b} - p'\, dotmu^{;b} \tag{2.1.8}$$

and taking the projection:

> temp3b := expand(P[a,-b]·temp3) : T(%);

$$PU\, P^{a}{}_{b}\, dotdu^{b} + dotPU\, P^{a}{}_{b}\, du^{b} = PU\, dotp'\theta P^{a}{}_{b}\, u^{b} + PU\, p'\theta P^{a}{}_{b}\, dotu^{b} + PU\, p'\, dottheta\, P^{a}{}_{b}\, u^{b} + dotPU\, p'\theta P^{a}{}_{b}\, u^{b} - dotp' P^{a}{}_{b}\,\mu^{;b} - p' P^{a}{}_{b}\, dotmu^{;b} \tag{2.1.9}$$

and elimination of terms:

> temp3c := expand(TEDS(P[a,-b]·u[b] = 0, temp3b)) : T(%);

$$PU\, P^{a}{}_{b}\, dotdu^{b} + dotPU\, P^{a}{}_{b}\, du^{b} = PU\, p'\theta P^{a}{}_{b}\, dotu^{b} - dotp' P^{a}{}_{b}\,\mu^{;b} - p' P^{a}{}_{b}\, dotmu^{;b} \tag{2.1.10}$$

Now we have a few substitutions:

```
> eq[30] : T(%);
```

$$dotmu + \theta\,(\mu + p) = 0 \tag{2.1.11}$$

```
> temp4 := eq[30] - op(2, (op(1, eq[30]))) : T(%);
```

$$dotmu = -\theta\,(\mu + p) \tag{2.1.12}$$

and also it can easily be shown, using the chain rule (see p30b notebook)

```
> temp5 := dotp =- `p'`·(mu + p)·theta : T(%);
```

$$dotp = -(\mu + p)\,p'\theta \tag{2.1.13}$$

and so

```
> temp6 := dotT(defPU) : T(%);
```

$$dotmu + dotp = dotPU \tag{2.1.14}$$

```
> temp7 := rhs(temp6) = lhs(temp6) : T(%);
```

$$dotPU = dotmu + dotp \tag{2.1.15}$$

```
> temp8 := factor(TEDS(temp4, temp7)) : T(%);
```

$$dotPU = -\mu\,\theta - p\,\theta + dotp \tag{2.1.16}$$

```
> temp9 := TEDS(defPU, factor(TEDS(temp5, temp8))) : T(%);
```

$$dotPU = -\theta\,(p' + 1)\,PU \tag{2.1.17}$$

and also it can easily be shown, using the chain rule (see p30b personal notebook): THIS STEP NEEDS CONFIRMATION

```
> temp10 := `dotp'`=- `p''`·PU·theta : T(%);
```

$$dotp' = -p''\,PU\,\theta \tag{2.1.18}$$

and so far we have

```
> temp11 := TEDS(temp10, TEDS(temp9, temp3c)) : T(%);
```

$$-P^{a}{}_{b}\,du^{b}\,\theta\,PU\,p' - P^{a}{}_{b}\,du^{b}\,\theta\,PU + PU\,P^{a}{}_{b}\,dotdu^{b} = PU\,p'\,\theta\,P^{a}{}_{b}\,dotu^{b} + P^{a}{}_{b}\,\mu^{;b}\,p''\,PU\,\theta - p'\,P^{a}{}_{b}\,dotmu^{;b} \tag{2.1.19}$$

```
> eq[30] : T(%);
```

$$dotmu + \theta\,(\mu + p) = 0 \tag{2.1.20}$$

Now we have shown in HC69:

```
> HC[69] := subs(B =-B, TEDS( -mu - p =-PU,
    collect(parse("dotmu[-B] = -theta[-B]*mu-theta[-B]*p-theta*mu[-B]-theta*`p'`*mu[-B]-
    mu[-D]*u[d,-B]"), [theta[-B]]))) : T(%);
```

$$dotmu^{;b} = -p'\,\theta\,\mu^{;b} - PU\,\theta^{;b} - \theta\,\mu^{;b} - \mu_{;d}\,u^{d\;;b} \tag{2.1.21}$$

and substituting in temp11:

```
> temp15a := expand(TEDS(HC[69], temp11)) : T(%);
```

$$-P^{a}{}_{b}\,du^{b}\,\theta\,PU p' - P^{a}{}_{b}\,du^{b}\,\theta\,PU + PU P^{a}{}_{b}\,dotdu^{b} = PU p'\,\theta\,P^{a}{}_{b}\,dotu^{b} + P^{a}{}_{b}\,\mu^{;b}\,p''\,PU\theta + p^{2}\,\theta\,P^{a}{}_{b}\,\mu^{;b} + PU p' P^{a}{}_{b}\,\theta^{;b} + p'\,\theta\,P^{a}{}_{b}\,\mu^{;b} + p' P^{a}{}_{b}\,\mu_{;d}\,u^{d\;;b} \tag{2.1.22}$$

> temp15a2 := expand(TEDS(P[a,-b]·u[b]=0, temp15a)) : T(%)

$$-P^{a}{}_{b}\,du^{b}\,\theta\,PU p' - P^{a}{}_{b}\,du^{b}\,\theta\,PU + PU P^{a}{}_{b}\,dotdu^{b} = PU p'\,\theta\,P^{a}{}_{b}\,dotu^{b} + P^{a}{}_{b}\,\mu^{;b}\,p''\,PU\theta + p^{2}\,\theta\,P^{a}{}_{b}\,\mu^{;b} + PU p' P^{a}{}_{b}\,\theta^{;b} + p'\,\theta\,P^{a}{}_{b}\,\mu^{;b} + p' P^{a}{}_{b}\,\mu_{;d}\,u^{d\;;b} \tag{2.1.23}$$

> temp15a3 := expand(TEDS(dotu[b]=du[b], temp15a2)) : T(%)

$$-P^{a}{}_{b}\,du^{b}\,\theta\,PU p' - P^{a}{}_{b}\,du^{b}\,\theta\,PU + PU P^{a}{}_{b}\,dotdu^{b} = P^{a}{}_{b}\,du^{b}\,\theta\,PU p' + P^{a}{}_{b}\,\mu^{;b}\,p''\,PU\theta + p^{2}\,\theta\,P^{a}{}_{b}\,\mu^{;b} + PU p' P^{a}{}_{b}\,\theta^{;b} + p'\,\theta\,P^{a}{}_{b}\,\mu^{;b} + p' P^{a}{}_{b}\,\mu_{;d}\,u^{d\;;b} \tag{2.1.24}$$

> temp15a4 := expand(\`p'\`·TEDS(P[a,-b]·du[b]=du[a], temp15a3)) : T(%)

$$-PU p^{2}\,\theta\,du^{a} - PU p'\,\theta\,du^{a} + PU p' P^{a}{}_{b}\,dotdu^{b} = PU p' p''\,\theta\,P^{a}{}_{b}\,\mu^{;b} + p^{3}\,\theta\,P^{a}{}_{b}\,\mu^{;b} + PU p^{2}\,\theta\,du^{a} + PU p^{2} P^{a}{}_{b}\,\theta^{;b} + p^{2}\,\theta\,P^{a}{}_{b}\,\mu^{;b} + p^{2} P^{a}{}_{b}\,\mu_{;d}\,u^{d\;;b} \tag{2.1.25}$$

> temp15a5 := isolate(temp15a4, \`p'\`·P[a,-b]·dotdu[b]·PU) : T(%);

$$PU p' P^{a}{}_{b}\,dotdu^{b} = PU p' p''\,\theta\,P^{a}{}_{b}\,\mu^{;b} + p^{3}\,\theta\,P^{a}{}_{b}\,\mu^{;b} + 2\,PU p^{2}\,\theta\,du^{a} + PU p^{2} P^{a}{}_{b}\,\theta^{;b} + p^{2}\,\theta\,P^{a}{}_{b}\,\mu^{;b} + p^{2} P^{a}{}_{b}\,\mu_{;d}\,u^{d\;;b} + PU p'\,\theta\,du^{a} \tag{2.1.26}$$

> #temp15a6:=Absorbg(TEDS(P[a,-b]=g[a,-b]+u[a]·u[-b], rhs(temp15a5))) : T(%);

> #temp15a7:=TEDS(mu[B]·u[-b]=-theta·PU, temp15a6) : T(%);

> #temp15a8:=collect(temp15a7, [theta, du[a], PU, \`p'\`]) : T(%);

so we have:

> $\frac{lhs(temp15a5)}{\text{`p'`}\cdot PU} = collect\left(expand\left(\frac{temp15a8}{\text{`p'`}\cdot PU}\right), [u[a], \text{theta}, du[a], \text{mu}[A]]\right)$: T(%);

$$P^{a}{}_{b}\,dotdu^{b} = \frac{temp15a8}{p'\,PU} \tag{2.1.27}$$

> eq55SSS : T(%);

$$P^{a}{}_{b}\,dotdu^{b} = \left(p' - \frac{1}{3} - \frac{(\mu + p)\,p''}{p'}\right)\theta\,du^{a} + \omega^{a}{}_{b}\,du^{b} + p' P^{a\,b}\,\theta_{;b} \tag{2.1.28}$$

so we are not really close here....let us relook at temp15a5

> $expand\left(\frac{\text{temp15a5}}{`p'`}\right)$: T(%);

$$PU\,P^{a}{}_{b}\,dotdu^{b} = P^{a}{}_{b}\,\mu^{;b}\,p''\,PU\,\theta + p^2\,\theta\,P^{a}{}_{b}\,\mu^{;b} + 2\,PU\,p'\,\theta\,du^{a} + PU\,p'\,P^{a}{}_{b}\,\theta^{;b} + p'\,\theta\,P^{a}{}_{b}\,\mu^{;b} + p'\,P^{a}{}_{b}\,\mu_{;d}\,u^{d\,;b} + \theta\,PU\,du^{a} \quad \textbf{(2.1.29)}$$

> temp15a51 := expand$\left(\frac{\text{TEDS}(\text{P[a,-b]}\cdot `p'`\cdot\text{mu}[B] = -du[a]\cdot PU, \text{temp15a5})}{`p'`\cdot PU}\right)$: $T(\%)$;

$$P^{a}{}_{b}\,dotdu^{b} = -\frac{PU\,p''\,\theta\,du^{a}}{p'} + p'\,\theta\,du^{a} + p'\,P^{a}{}_{b}\,\theta^{;b} + \frac{p'\,P^{a}{}_{b}\,\mu_{;d}\,u^{d\,;b}}{PU} \quad \textbf{(2.1.30)}$$

> temp15a52 := collect(temp15a51, [theta, du[a]]) : $T(\%)$;

$$P^{a}{}_{b}\,dotdu^{b} = \left(-\frac{PU\,p''}{p'} + p'\right) du^{a}\,\theta + p'\,P^{a}{}_{b}\,\theta^{;b} + \frac{p'\,P^{a}{}_{b}\,\mu_{;d}\,u^{d\,;b}}{PU} \quad \textbf{(2.1.31)}$$

Let us look at the 2nd term on the RHS of temp15a52

> temp15a53 := $\frac{\text{P[a,-b]}\cdot\text{`p'`}\cdot\text{mu[-D]}\cdot u[d, B]}{PU}$: $T(\%)$;

$$\frac{p'\,P^{a}{}_{b}\,\mu_{;d}\,u^{d\,;b}}{PU} \quad \textbf{(2.1.32)}$$

Now

> $aside := TEDS(p[-B] = `p'`\cdot\text{mu}[-B], TEDS(p + \text{mu} = PU, eq[31]))$: $T(\%)$;

$$P^{a\,b}\,p'\,\mu_{;b} + du^{a}\,PU = 0 \quad \textbf{(2.1.33)}$$

> $aside2 := Absorbg(TEDS(P[a, b] = g[a, b] + u[a]\cdot u[b], aside))$: $T(\%)$;

0, "not a tensor"

$$p'\,\mu_{;b}\,u^{a}\,u^{b} + du^{a}\,PU + p'\,\mu^{;a} = 0 \quad \textbf{(2.1.34)}$$

> $aside3 := TEDS(\text{mu}[-B]\cdot u[b] = -PU\cdot\text{theta}, aside2)$: $T(\%)$;

$$-p'\,u^{a}\,PU\,\theta + du^{a}\,PU + p'\,\mu^{;a} = 0 \quad \textbf{(2.1.35)}$$

> $aside4 := subs(a = -d, A = -\text{D}, isolate(aside3, `p'`\cdot\text{mu}[A]))$: $T(\%)$;

$$p'\,\mu_{;d} = PU\,p'\,\theta\,u_{d} - PU\,du_{d} \quad \textbf{(2.1.36)}$$

also from eq6:

> $aside5 := TEDS(\text{sigma}[d, b] = 0, subs(a = -d, A = -\text{D}, b = -b, B = -B, eq[6]))$: $T(\%)$;

$$u^{d\,;b} = \frac{1}{3}\,\theta\,P^{d\,b} + \omega^{d\,b} - du^{d}\,u^{b} \quad \textbf{(2.1.37)}$$

and so temp15a53

> $aside6 := expand\left(\frac{expand(P[a,-b]\cdot rhs(aside4)\cdot rhs(aside5))}{PU}\right)$: $T(\%)$;

$$\frac{1}{3}\,P^{a}{}_{b}\,p'\,\theta^2\,P^{d\,b}\,u_{d} - \frac{1}{3}\,P^{a}{}_{b}\,\theta\,P^{d\,b}\,du_{d} + P^{a}{}_{b}\,p'\,\theta\,\omega^{d\,b}\,u_{d} - P^{a}{}_{b}\,du_{d}\,\omega^{d\,b} \quad \textbf{(2.1.38)}$$

$$-P^{a}{}_{b}\,p'\,\theta\,du^{d}\,u^{b}\,u_{d}+P^{a}{}_{b}\,du^{d}\,du_{d}\,u^{b}$$

> aside7 := expand(TEDS(P[d, b]·u[-d]=0, aside6)) : T(%);

$$-\frac{1}{3}\,P^{a}{}_{b}\,\theta\,P^{d\,b}\,du_{d}+P^{a}{}_{b}\,p'\,\theta\,\omega^{d\,b}\,u_{d}-P^{a}{}_{b}\,du_{d}\,\omega^{d\,b}-P^{a}{}_{b}\,p'\,\theta\,du^{d}\,u^{b}\,u_{d}+P^{a}{}_{b}\,du^{d}\,du_{d}\,u^{b} \quad \textbf{(2.1.39)}$$

> aside8 := expand(TEDS(omega[d, b]·u[-d]=0, aside7)) : T(%);

$$-\frac{1}{3}\,P^{a}{}_{b}\,\theta\,P^{d\,b}\,du_{d}-P^{a}{}_{b}\,du_{d}\,\omega^{d\,b}-P^{a}{}_{b}\,p'\,\theta\,du^{d}\,u^{b}\,u_{d}+P^{a}{}_{b}\,du^{d}\,du_{d}\,u^{b} \quad \textbf{(2.1.40)}$$

> aside9 := expand(TEDS(P[a,-b]·u[b]=0, aside8)) : T(%);

$$-\frac{1}{3}\,P^{a}{}_{b}\,\theta\,P^{d\,b}\,du_{d}-P^{a}{}_{b}\,du_{d}\,\omega^{d\,b} \quad \textbf{(2.1.41)}$$

> aside10 := expand(TEDS(P[a,-b]·du[-d]·P[d, b]=du[a], aside9)) : T(%);

$$-P^{a}{}_{b}\,du_{d}\,\omega^{d\,b}-\frac{1}{3}\,\theta\,du^{a} \quad \textbf{(2.1.42)}$$

> aside11 := expand(TEDS(P[a,-b]·omega[d, b]=-omega[a, d], aside10)) : T(%);

$$du_{d}\,\omega^{a\,d}-\frac{1}{3}\,\theta\,du^{a} \quad \textbf{(2.1.43)}$$

and so we have

> aside12 := temp15a53=aside11 : T(%);

$$\frac{p'\,P^{a}{}_{b}\,\mu_{;d}\,u^{d\,;b}}{PU}=du_{d}\,\omega^{a\,d}-\frac{1}{3}\,\theta\,du^{a} \quad \textbf{(2.1.44)}$$

> collect(TEDS(aside12, temp15a52), [theta, du[a]]) : T(%);

$$P^{a}{}_{b}\,dotdu^{b}=\left(-\frac{PU\,p''}{p'}+p'-\frac{1}{3}\right)du^{a}\,\theta+p'\,P^{a}{}_{b}\,\theta^{;b}+du_{d}\,\omega^{a\,d} \quad \textbf{(2.1.45)}$$

> eq55SSS : T(%);

$$P^{a}{}_{b}\,dotdu^{b}=\left(p'-\frac{1}{3}-\frac{(\mu+p)\,p''}{p'}\right)\theta\,du^{a}+\omega^{a}{}_{b}\,du^{b}+p'\,P^{a\,b}\,\theta_{;b} \quad \textbf{(2.1.46)}$$

and so we are done....

> convert(eq55SSS, string);

"P[a,-b]*dotdu[b] = (`p'`-1/3-(mu+p)*`p''`/`p'`)*theta*du[a]+omega[a,-b]*du[b]+`p'`*P[a, b]*theta[-B]" **(2.1.47)**